Solutions to Manfredo P. do Carmo's Differential Geometry of Curves and Surfaces

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Preface

This is a solution manual for the textbook which used for "Introduction to Differential Geometry" in Seoul National University. I hope these solutions are helpful for people who loves mathematics. The solutions may contain mathematical errors, so I recommend reading them critically. If you find any mathematical errors or typos, please report them to qwerty12021@snu.ac.kr. Thank you for your valuable feedback.

This file is uploaded to 🌐https://geniuslhs.com/solutions/do-carmo-differential-geometry.pdf. I recommend checking it regularly, as it may be updated.

For clarity and consistency, the following basic notations have been modified from the original text: the set of real numbers R is denoted as $\mathbb{R}$, the unit sphere S^2 is denoted as $\mathbb{S}^2$, vectors are written in bold (e.g. p is written as $\mathbf{p}$), the set notation $\{x \, ; P(x)\}$ has been changed to $\{x \mid P(x)\}$, the restriction of a function $f|A$ is denoted as $f|_A$, the fundamental forms $I_p(\mathbf{v})$ and $II_p(\mathbf{v})$ are denoted as $\mathbf{I}_p(\mathbf{v})$ and $\mathbf{II}_p(\mathbf{v})$ to avoid confusion with the interval I.

The following common conventions were considered but not applied in this manual: the curvature k as κ and the tangent space $T_p(S)$ as T_pS.

The solutions are constructed strictly upon fundamental definitions and theorems, aiming for a balance between conciseness and conceptual depth. Where applicable, graphs and diagrams have been integrated to provide intuitive visual support. To help you navigate the content, the following tags are used:

- **Note** Contains supplementary information and references.
- Unsolved Indicates problems that remain unsolved or only partially completed.

Contents

History

- [2026.02.28.] First release.

1-2. Parametrized Curves

1. $\alpha(t) = (\sin t, \cos t)$ $(0 \le t \le 2\pi)$

2. Since $|\alpha(t)|^2$ is at a minimum at $t = t_0$, its derivative

$$\frac{d}{dt}(|\alpha(t)|^2) = \frac{d}{dt}(\alpha(t) \cdot \alpha(t)) = 2\alpha(t) \cdot \alpha'(t)$$

is equal to 0 at $t = t_0$. Hence $\alpha(t_0) \cdot \alpha'(t_0) = 0$ and the two vectors are orthogonal.

3. For a parametrized curve $\alpha : I \to \mathbb{R}^3$, let $\alpha(t) = (x(t), y(t), z(t))$. Then the condition $\alpha''(t) = 0$ implies that $x''(t) = y''(t) = z''(t) = 0$. Integrating both sides twice to get

$$x(t) = x_0 + x_1 t, \quad y(t) = y_0 + y_1 t, \quad z(t) = z_0 + z_1 t$$

for some constants $x_0, x_1, y_0, y_1, z_0, z_1 \in \mathbb{R}$. Therefore, we obtain

$$\alpha(t) = (x(t), y(t), z(t)) = (x_0, y_0, z_0) + (x_1, y_1, z_1)t$$

and α must be a straight line.

4. Since $\alpha'(t)$ is orthogonal to $\mathbf{v}$ for all $t \in I$, we have $\alpha'(t) \cdot \mathbf{v} = 0$ for all $t \in I$. Also, $\alpha(0)$ is orthogonal to $\mathbf{v}$, so $\alpha(0) \cdot \mathbf{v} = 0$. Now define $f(t) = \alpha(t) \cdot \mathbf{v}$. Then $f'(t) = \alpha'(t) \cdot \mathbf{v} = 0$ for all $t \in I$ and $f(0) = \alpha(0) \cdot \mathbf{v} = 0$; therefore, $f(t) = 0$ for all $t \in I$. This implies that $\alpha(t)$ is orthogonal to $\mathbf{v}$ for all $t \in I$.

5. First, suppose that $|\alpha(t)|$ is a nonzero constant. Then $|\alpha(t)|^2 = \alpha(t) \cdot \alpha(t)$ is also a nonzero constant, so we have $(\alpha(t) \cdot \alpha(t))' = 2\alpha(t) \cdot \alpha'(t) = 0$ for all $t \in I$. In other words, $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

Conversely, suppose that $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$, i.e., $\alpha(t) \cdot \alpha'(t) = 0$. Then

$$\frac{d}{dt}(|\alpha(t)|^2) = (\alpha(t) \cdot \alpha(t))' = 2\alpha(t) \cdot \alpha'(t) = 0 \quad (t \in I)$$

which implies that $|\alpha(t)|^2$ is a constant, so $|\alpha(t)|$ is also a constant. Note that $|\alpha(t)| \ne 0$. Otherwise, α reduces to a constant point, implying $\alpha'(t) = 0$, which violates the assumption that $\alpha'(t) \ne 0$.
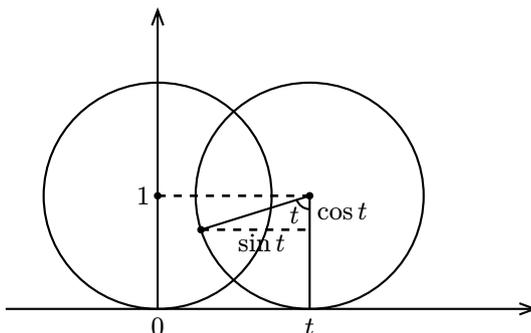

1-3. Regular Curves; Arc Length

1. The tangent lines to the curve $\alpha(t) = (3t, 3t^2, 2t^3)$ are parallel to $\alpha'(t) = (3, 6t, 6t^2)$, and the line $y = 0, z = x$ is parallel to a vector $(1, 0, 1)$. Therefore the angle $\theta(t)$ between two vectors satisfies

$$\cos \theta(t) = \frac{\alpha'(t) \cdot (1, 0, 1)}{|\alpha'(t)||(1, 0, 1)|} = \frac{3 + 6t^2}{\sqrt{9 + 36t^2 + 36t^4}\sqrt{2}} = \frac{1}{\sqrt{2}}$$

and $\theta(t) = \pi/4$ is constant.

2.

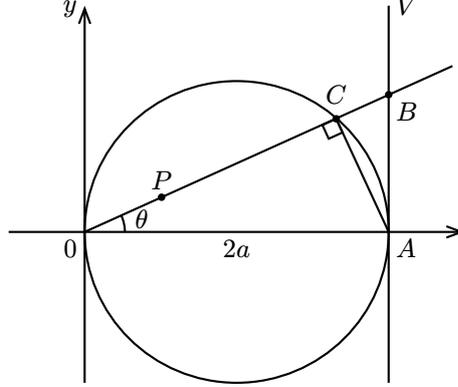**a.** By interpreting the situation above, the parametrized curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ is given by $\alpha(t) = (t - \sin t, 1 - \cos t)$. The singular points of α are the points t where $\alpha'(t) = (1 - \cos t, \sin t) = (0, 0)$, which are $t = 2n\pi$ $(n \in \mathbb{Z})$.

b. Observe that the disk rotates exactly once when t increases from 0 to 2π. Therefore the arc length is

$$\int_0^{2\pi} |\alpha'(t)|\, dt = \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\, dt = \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$$

$$= 2\int_0^{2\pi} \sqrt{\frac{1 - \cos t}{2}}\, dt = 2\int_0^{2\pi} \sin\frac{t}{2}\, dt = 2\left[-2\cos\frac{t}{2}\right]_0^{2\pi} = 8.$$

3.

a. Since $0B = 2a\sec\theta$ and $0C = 2a\cos\theta$, we get $CB = 2a(\sec\theta - \cos\theta)$. Then $0P = CB$ implies $P = 2a(\sec\theta - \cos\theta)(\cos\theta, \sin\theta)$. Let $t = \tan\theta$, then

$$2a(\sec\theta - \cos\theta)\cos\theta = 2a(1 - \cos^2\theta) = 2a\left(1 - \frac{1}{1+t^2}\right) = \frac{2at^2}{1+t^2},$$

$$2a(\sec\theta - \cos\theta)\sin\theta = \frac{2at^2}{1+t^2}\cdot\tan\theta = \frac{2at^3}{1+t^2}.$$

Therefore the parametrized curve is given by

$$\alpha(t) = \left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2}\right), \quad t \in \mathbb{R}.$$

b. Since $\alpha'(t) = \left(\frac{4at}{(1+t^2)^2}, \frac{2a(3t^2+t^4)}{(1+t^2)^2}\right)$, we have $\alpha'(0) = (0,0)$. Therefore the origin $(0,0)$ is a singular point of the cissoid.

c. Observe that

$$\lim_{t\to\infty}\alpha(t) = (2a, \infty), \quad \lim_{t\to\infty}\alpha'(t) = (0, 2a).$$

Thus the curve and its tangent approach the line $x = 2a$ as $t \to \infty$.

4. a. The tangent vector of $\alpha(t)$ is

$$\alpha'(t) = \left(\cos t, -\sin t + \frac{\sec^2(t/2)}{2\tan(t/2)}\right) = \left(\cos t, -\sin t + \frac{1}{2\cos(t/2)\sin(t/2)}\right)$$

$$= \left(\cos t, \frac{1}{\sin t} - \sin t\right) = \left(\cos t, \frac{\cos^2 t}{\sin t}\right).$$

Therefore $\alpha'(t) = (0,0)$ if and only if $\cos t = 0$, that is, $t = \pi/2$.

b.
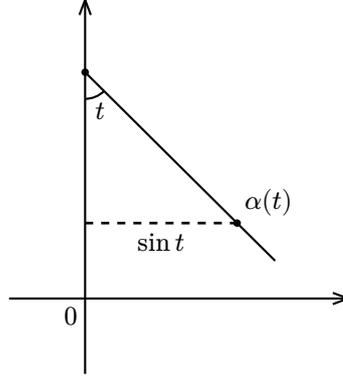
Since $\alpha'(t) \parallel (\sin t, \cos t)$, the angle between the tangent line and y axis is t. Moreover, the x-coordinate of $\alpha(t)$ is $\sin t$, so the length of the specified segment is always 1.

5. a. The tangent vector of $\alpha(t)$ is $\alpha'(t) = 3a\left(\frac{1-2t^3}{(1+t^3)^2}, \frac{2t-t^4}{(1+t^3)^2}\right)$, so $\alpha'(0) = (3a, 0)$. Hence α is tangent to the x axis for $t = 0$.

b. As $t \to \infty$, $\alpha(t) \to (0,0)$ and $\alpha'(t) \to (0,0)$ by the equation above.

c. Let $\alpha(t) = (x(t), y(t))$. Then

$$\lim_{t \to -1} \frac{y'(t)}{x'(t)} = \lim_{t \to -1} \frac{2t - t^4}{1 - 2t^3} = -1.$$

Therefore the slope of the tangent line approaches -1. Also,

$$\lim_{t \to -1} x(t) + y(t) = \lim_{t \to -1} \frac{3a(t + t^2)}{1 + t^3} = \lim_{t \to -1} \frac{3at}{1 - t + t^2} = -a$$

implies that the curve and its tangent approach the line $x + y + a = 0$.

> **Note** To be more precise, we have to define what "approach the line" means. Every line in the plane can be expressed by the equation $ax + by + c = 0$, where either $a \neq 0$ or $b \neq 0$. Note that multiplying the vector (a, b, c) by a nonzero constant does not change the line, so the line does not determine the vector (a, b, c) uniquely. However, there exists only two normal vectors (a_0, b_0) such that (a_0, b_0, c_0) expresses the given line. Now we say that a family of the lines
>
> $$L(t) = \{(x, y) \in \mathbb{R}^2 \mid a(t)x + b(t)y + c(t) = 0\}$$
>
> approach the line
>
> $$L_0 = \{(x, y) \in \mathbb{R}^2 \mid a_0 x + b_0 y + c_0 = 0\}, \quad \text{where } a_0^2 + b_0^2 = 1$$
>
> as $t \to t_0$ if there exists the choice of signs $\mu(t) = \pm 1$ such that
>
> $$\lim_{t \to t_0} \mu(t) \frac{(a(t), b(t), c(t))}{\sqrt{a(t)^2 + b(t)^2}} = (a_0, b_0, c_0).$$

According to this definition, we can explain the behavior of the tangent line more strictly. Let $\mathbf{n}(t) = 3a\left(\frac{-2t+t^4}{(1+t^3)^2}, \frac{1-2t^3}{(1+t^3)^2}\right)$ be a vector orthogonal to $\alpha'(t)$. Then the tangent line of $\alpha(t)$ is given by $\mathbf{n}(t) \cdot ((x, y) - \alpha(t)) = 0$, that is,

$$\frac{-2t + t^4}{(1 + t^3)^2}\left(x - \frac{3at}{1 + t^3}\right) + \frac{1 - 2t^3}{(1 + t^3)^2}\left(y - \frac{3at^2}{1 + t^3}\right) = 0$$

$$\iff (-6at + 3at^4)x + (3a - 6at^3)y + 9a^2t^2 = 0.$$

Since $a(t) \to 9a, b(t) \to 9a, c(t) \to 9a^2$ as $t \to 0$, we get

$$\lim_{t \to 0} 1 \cdot \frac{a(t), b(t), c(t)}{\sqrt{a(t)^2 + b(t)^2}} = \left(\frac{1}{\sqrt{a^2 + 2}}, \frac{1}{\sqrt{a^2 + 2}}, \frac{a}{\sqrt{a^2 + 2}}\right).$$

By the definition above, the tangent line approaches to the line $\frac{1}{\sqrt{a^2+2}}x + \frac{1}{\sqrt{a^2+2}}y + \frac{a}{\sqrt{a^2+2}} = 0$, which is the same as $x + y + a = 0$.

6. a. Obviously $\alpha(t) \to (0,0)$ as $t \to \infty$. $\alpha(t)$ approaches the origin 0 spiraling around it because of the factor $(\cos t, \sin t)$.

b. $\alpha'(t) = a\big(be^{bt}\cos t - e^{bt}\sin t, be^{bt}\sin t + e^{bt}\cos t\big)$, so $\alpha'(t) \to (0,0)$ as $t \to \infty$. Since

$$|\alpha'(t)| = \sqrt{a^2\big((be^{bt}\cos t - e^{bt}\sin t)^2 + (be^{bt}\sin t + e^{bt}\cos t)^2\big)} = a\sqrt{b^2+1}\,e^{bt},$$

the given integral is

$$\lim_{t\to\infty}\int_{t_0}^{t}|\alpha'(t)|\,dt = \lim_{t\to\infty}\int_{t_0}^{t}a\sqrt{b^2+1}\,e^{bt}\,dt = a\sqrt{b^2+1}\lim_{t\to\infty}\left[\frac{e^{bt}}{b}\right]_{t_0}^{t} = -\frac{a}{b}\sqrt{b^2+1}\,e^{bt_0},$$

which is finite. Hence α has finite arc length on $[t_0, \infty)$.

7. a. First consider the line determined by $\alpha(h) = (h^3, h^2)$ and $\alpha(0) = (0,0)$. This line is given by $x - hy = 0$, or $\frac{1}{\sqrt{1+h^2}}x - \frac{h}{\sqrt{1+h^2}}y = 0$. This line has a limit position $x = 0$ as $h \to 0$, therefore $\alpha(t)$ has a weak tangent at $t = 0$.

Now consider the line determined by $\alpha(h) = (h^3, h^2)$ and $\alpha(k) = (k^3, k^2)$. This line can be expressed by

$$\frac{x - h^3}{k^3 - h^3} = \frac{y - h^2}{k^2 - h^2} \iff (k^2 - h^2)(x - h^3) = (k^3 - h^3)(y - h^2)$$

$$\iff (k+h)(x - h^3) = (k^2 + kh + h^2)(y - h^2)$$

$$\iff (k+h)x - (k^2 + kh + h^2)y + k^2h^2 = 0.$$

Hence $\alpha(t)$ has a strong tangent at $t = 0$ if and only if the unit vector

$$\frac{1}{\sqrt{(k+h)^2 + (k^2 + kh + h^2)^2}}(k+h, k^2 + kh + h^2)$$

converges in $\mathbb{R}^2$ as $(k,h) \to (0,0)$. However, this limit does not exist; the expression approaches $(0,-1)$ along the path $k = -h \to 0$, whereas it approaches $(1,0)$ along the path $k = h \to 0$.

b. The direction of the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0 + k)$ is $\frac{\alpha(t_0+h)-\alpha(t_0+k)}{h-k}$. Since $\alpha(t) = (x(t), y(t))$ is differentiable (that is, has derivative of first order), by mean value theorem, there exists $|\xi_1|, |\xi_2| < \max\{|h|, |k|\}$ such that

$$\frac{x(t_0 + h) - x(t_0 + k)}{h - k} = x'(t_0 + \xi_1), \quad \frac{y(t_0 + h) - y(t_0 + k)}{h - k} = y'(t_0 + \xi_2).$$

Also, α is of class C^1, so $\alpha'(t)$ is continuous and the direction vector converges to $(x'(t_0), y'(t_0)) = \alpha'(t_0)$ as $h, k \to 0$. It is trivial that α has a strong tangent at $t = t_0$.

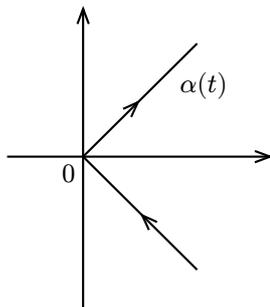**c.** The derivative of $\alpha(t) = (x(t), y(t))$ is

$$\alpha'(t) = \begin{cases} (2t, 2t), & (t \geq 0) \\ (2t, -2t), & (t < 0) \end{cases}.$$

Note that $\alpha'(t)$ is continuous, so α is of class C^1. However,

$$y'(t) = \begin{cases} 2t, & (t \geq 0) \\ -2t, & (t < 0) \end{cases}$$

is not differentiable at $t = 0$, so α is not of class C^2.

8. First observe that

$$l(\alpha, P) = \sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})| = \sum_{i=1}^{n} \left| \int_{t_{i-1}}^{t_i} \alpha'(t)\,dt \right| \le \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |\alpha'(t)|\,dt = \int_a^b |\alpha'(t)|\,dt.$$

On the other hand, since α' is uniformly continuous in $[a,b]$, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $t, s \in [a,b]$ with $|t - s| < \delta$, then

$$|\alpha'(t) - \alpha'(s)| < \frac{\varepsilon}{4(b-a)}.$$

Now suppose $|P| < \delta$. Then for every $t \in [t_{i-1}, t_i]$, we have $|\alpha'(t)| \le |\alpha'(t_i)| + \varepsilon$. Therefore

$$\int_{t_{i-1}}^{t_i} |\alpha'(t)|\,dt \le |\alpha'(t_i)|(t_i - t_{i-1}) + \frac{\varepsilon(t_i - t_{i-1})}{4(b-a)}$$

$$= \left| \int_{t_{i-1}}^{t_i} (\alpha'(t) + \alpha'(t_i) - \alpha'(t))\,dt \right| + \frac{\varepsilon(t_i - t_{i-1})}{4(b-a)}$$

$$\le \left| \int_{t_{i-1}}^{t_i} \alpha'(t)\,dt \right| + \left| \int_{t_{i-1}}^{t_i} (\alpha'(t_i) - \alpha'(t))\,dt \right| + \frac{\varepsilon(t_i - t_{i-1})}{4(b-a)}$$

$$\le |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\varepsilon(t_i - t_{i-1})}{2(b-a)}.$$

If we add these inequalities, we obtain

$$\int_a^b |\alpha'(t)|\,dt \le l(\alpha, P) + \frac{\varepsilon}{2} < l(\alpha, P) + \varepsilon.$$

Hence we conclude that

$$\left| \int_a^b |\alpha'(t)|\,dt - l(\alpha, P) \right| < \varepsilon.$$

9. **a.** See the definition of $l(\alpha, P)$ in Exercise 8. Observe that as the partition becomes finer and finer, the polygon approaches the curve more and more closely, and $l(\alpha, P)$ becomes larger and larger by triangle inequality. Therefore we can define the arc length of α by

$$\sup_P l(\alpha, P),$$

where the supremum is taken over all partitions of $[a,b]$.

b. Observe that the curve $\alpha(t) = (t, t\sin(\pi/t))$ passes through the points

$$\left(\frac{1}{n}, 0 \right), \quad \left(\frac{1}{n+1/2}, \frac{(-1)^n}{n+1/2} \right), \quad \left(\frac{1}{n+1}, 0 \right)$$

for every $n \in \mathbb{N}$. Hence the arc length of the portion of the curve corresponding to $1/(n+1) \le t \le 1/n$ is at least $2/(n+1/2)$. Then the length of the curve in the interval $1/N \le t \le 1$ is greater than

$$\sum_{n=1}^{N-1}\left(\text{length for } \frac{1}{n+1} \leq t \leq \frac{1}{n}\right) \geq \sum_{n=1}^{N-1}\frac{2}{n+1/2} > 2\sum_{n=1}^{N-1}\frac{1}{n+1}.$$

Since this sum tends to infinity as $N \to \infty$, arc length of α in a closed interval $[0, 1]$ is unbounded.

10. a. By the Cauchy–Schwarz inequality, we have $\alpha'(t) \cdot \mathbf{v} \leq |\alpha'(t)|$. Therefore

$$(\mathbf{q} - \mathbf{p}) \cdot \mathbf{v} = \int_a^b \alpha'(t) \cdot \mathbf{v} \, dt \leq \int_a^b |\alpha'(t)| \, dt.$$

b. Let $\mathbf{v} = (\mathbf{q} - \mathbf{p})/|\mathbf{q} - \mathbf{p}|$, then $(\mathbf{q} - \mathbf{p}) \cdot \mathbf{v} = |\mathbf{q} - \mathbf{p}|$, so

$$|\alpha(b) - \alpha(a)| = |\mathbf{q} - \mathbf{p}| \leq \int_a^b |\alpha'(t)| \, dt.$$

1-4. The Vector Product in $\mathbb{R}^3$

1. a. The basis is negative;

$$\left|\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\right| = -4 < 0.$$

b. The basis is positive;

$$\left|\begin{pmatrix} 1 & 3 & 5 \\ 2 & 3 & 7 \\ 4 & 8 & 3 \end{pmatrix}\right| = 39 > 0.$$

2. To prove that $\mathbf{v} = (a, b, c)$ is perpendicular to the plane, we have to show that every line contained in the plane P is perpendicular to $\mathbf{v}$. Now select any line l in P and choose $(x_1, y_1, z_1), (x_2, y_2, z_2) \in l$ so that $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ to be the direction of line l. Since $(x_1, y_1, z_1), (x_2, y_2, z_2) \in P$, we have $ax_1 + by_1 + cz_1 + d = ax_2 + by_2 + cz_2 + d = 0$ and

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0 \implies \mathbf{v} \cdot (x_2 - x_1, y_2 - y_1, z_2 - z_1) = 0.$$

Therefore $\mathbf{v} \perp l$ for any line $l \in P$, and $\mathbf{v}$ is perpendicular to the plane P.

Let (x, y, z) be a point in P, so we have $ax + by + cz = 0$. By Cauchy–Schwarz inequality,

$$\sqrt{x^2 + y^2 + z^2} \geq \frac{|ax + by + cz|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Hence the distance from the plane to the origin is $|d|/\sqrt{a^2 + b^2 + c^2}$.

3. The angle of intersection of the two planes $5x + 3y + 2x - 4 = 0$ and $3x + 4y - 7x = 0$ is equal to the angle between the two normal vectors $\mathbf{n}_1 = (5, 3, 2)$ and $\mathbf{n}_2 = (3, 4, -7)$. Therefore

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{13}{2\sqrt{703}}, \quad \theta = \arccos\left(\frac{13}{2\sqrt{703}}\right).$$

4. Two planes are parallel if and only if two normal vectors are parallel. Hence there exists k such that

$$(a_1, b_1, c_1) = k(a_2, b_2, c_2) \implies \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = k,$$

where the convention is made that if a denominator is zero, the corresponding numerator is also zero.

5. Since $\mathbf{a} \wedge \mathbf{b}$ are perpendicular to both $\mathbf{a}$ and $\mathbf{b}$,

$$(\mathbf{p} - \mathbf{p}_1) \wedge (\mathbf{p} - \mathbf{p}_2) \cdot (\mathbf{p} - \mathbf{p}_3) = 0$$
$$\iff \{(\mathbf{p} - \mathbf{p}_1) \wedge (\mathbf{p} - \mathbf{p}_2) + (\mathbf{p}_3 - \mathbf{p}) \wedge (\mathbf{p} - \mathbf{p}_2) + (\mathbf{p}_3 - \mathbf{p}_1) \wedge (\mathbf{p}_3 - \mathbf{p})\} \cdot (\mathbf{p} - \mathbf{p}_3) = 0$$
$$\iff (\mathbf{p}_3 - \mathbf{p}_1) \wedge (\mathbf{p}_3 - \mathbf{p}_2) \cdot (\mathbf{p} - \mathbf{p}_3) = 0.$$

Note that the normal vector of the plane passing through the three points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ is parallel to $(\mathbf{p}_3 - \mathbf{p}_1) \wedge (\mathbf{p}_3 - \mathbf{p}_2)$. Hence $\mathbf{p}$ is a point of the plane if and only if $(\mathbf{p}_3 - \mathbf{p}_1) \wedge (\mathbf{p}_3 - \mathbf{p}_2) \cdot (\mathbf{p} - \mathbf{p}_3) = 0$, that is, $(\mathbf{p} - \mathbf{p}_1) \wedge (\mathbf{p} - \mathbf{p}_2) \cdot (\mathbf{p} - \mathbf{p}_3) = 0$.

6. Since the line of intersection l is contained in both planes, l is perpendicular to the normal vectors $\mathbf{v}_1, \mathbf{v}_2$. Therefore l is parallel to the vector $\mathbf{u} = \mathbf{v}_1 \wedge \mathbf{v}_2$. Now suppose $(x, y, z) \in l$, then $(x - x_0, y - y_0, z - z_0)$ is parallel to l, therefore

$$(x - x_0, y - y_0, z - z_0) = \mathbf{u}t \implies x - x_0 = u_1 t, \ y - y_0 = u_2 t, \ z - z_0 = u_3 t.$$

7. A plane and a line are parallel when the normal vector to the plane is perpendicular to the direction of the line, that is, $(a, b, c) \cdot (u_1, u_2, u_3) = 0 \iff au_1 + bu_2 + cu_3 = 0$.
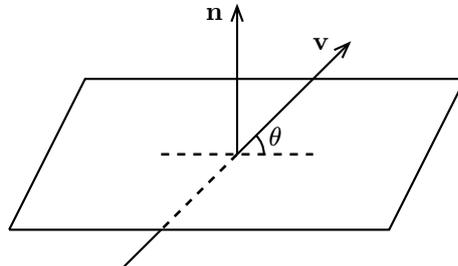
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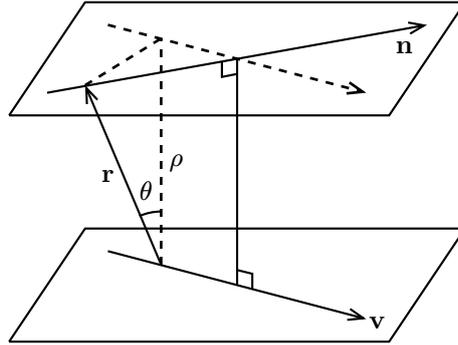
The distance ρ between the nonparallel lines is given by $|\mathbf{r}||\cos\theta|$, where θ is the angle between $\mathbf{r}$ and the normal vector $\mathbf{u} \wedge \mathbf{v}$. Therefore

$$\rho = |\mathbf{r}||\cos\theta| = |\mathbf{r}|\frac{|(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{r}|}{|\mathbf{u} \wedge \mathbf{v}||\mathbf{r}|} = \frac{|(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{r}|}{|\mathbf{u} \wedge \mathbf{v}|}.$$

9.

Let θ be the angle of intersection of the plane and the line. Then the angle between the normal vector $\mathbf{n} = (3, 4, 7)$ and direction $(3, 5, 9)$ is $\pi/2 - \theta$. Therefore

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) = \frac{\mathbf{n} \cdot \mathbf{v}}{|\mathbf{n}||\mathbf{v}|} = \frac{92}{\sqrt{74}\sqrt{115}} \implies \theta = \arcsin\left(\frac{92}{\sqrt{8510}}\right).$$

10. It is easy to see that

$$\begin{pmatrix} \mathbf{u} \cdot \mathbf{u} & \mathbf{u} \cdot \mathbf{v} \\ \mathbf{v} \cdot \mathbf{u} & \mathbf{v} \cdot \mathbf{v} \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}.$$

Now let $\mathbf{u}' = (u_1, u_2, 0), \mathbf{v}' = (v_1, v_2, 0) \in \mathbb{R}^3$. Then $A = |\mathbf{u}' \wedge \mathbf{v}'|$ so

$$A^2 = |\mathbf{u}' \wedge \mathbf{v}'|^2 = \left|\begin{pmatrix} \mathbf{u}' \cdot \mathbf{u}' & \mathbf{u}' \cdot \mathbf{v}' \\ \mathbf{v}' \cdot \mathbf{u}' & \mathbf{v}' \cdot \mathbf{v}' \end{pmatrix}\right| = \left|\begin{pmatrix} \mathbf{u} \cdot \mathbf{u} & \mathbf{u} \cdot \mathbf{v} \\ \mathbf{v} \cdot \mathbf{u} & \mathbf{v} \cdot \mathbf{v} \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}\right| = \left|\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\right|\left|\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}\right| = \left|\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\right|^2.$$

11. a.

The volume of a parallelepiped is given by (area of the base) · (height). Let θ be the angle between $\mathbf{w}$ and $\mathbf{u} \wedge \mathbf{v}$. Since the area of the base is $|\mathbf{u} \wedge \mathbf{v}|$ and height is $|\mathbf{w}||\cos\theta|$, we have

$$V = |\mathbf{u} \wedge \mathbf{v}||\mathbf{w}||\cos\theta| = |(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w}|.$$

Now we can define oriented volume $(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w}$, which is positive if and only if the basis $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is positive.

b. Since $|(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w}| = \det(\mathbf{u}, \mathbf{v}, \mathbf{w})$, we have

$$V^2 = |(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w}|^2 = \det(\mathbf{u}, \mathbf{v}, \mathbf{w})^2 = \left|\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix}\right|^2$$

$$= \left|\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix}\right|\left|\begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix}\right| = \left|\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix}\begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix}\right|$$

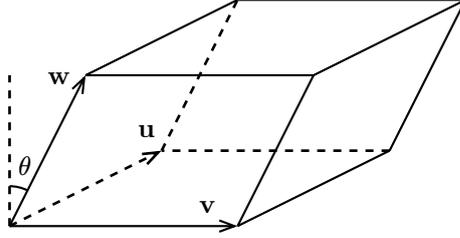$$= \left|\begin{pmatrix} \mathbf{u} \cdot \mathbf{u} & \mathbf{u} \cdot \mathbf{v} & \mathbf{u} \cdot \mathbf{w} \\ \mathbf{v} \cdot \mathbf{u} & \mathbf{v} \cdot \mathbf{v} & \mathbf{v} \cdot \mathbf{w} \\ \mathbf{w} \cdot \mathbf{u} & \mathbf{w} \cdot \mathbf{v} & \mathbf{w} \cdot \mathbf{w} \end{pmatrix}\right|.$$

12. If $\mathbf{u} \wedge \mathbf{v} = \mathbf{w}$, then $\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot (\mathbf{u} \wedge \mathbf{v}) = 0$ so $\mathbf{v}$ is perpendicular to $\mathbf{w}$.

Conversely, suppose that $\mathbf{v}$ is perpendicular to $\mathbf{w}$, so $\mathbf{v} \cdot \mathbf{w} = 0$. Let $\mathbf{u}_0 = \frac{1}{|\mathbf{v}|^2}(\mathbf{v} \wedge \mathbf{w})$ then

$$\mathbf{u}_0 \wedge \mathbf{v} = \frac{1}{|\mathbf{v}|^2}((\mathbf{v} \wedge \mathbf{w}) \wedge \mathbf{v}) = \frac{1}{|\mathbf{v}|^2}((\mathbf{v} \cdot \mathbf{v})\mathbf{w} - (\mathbf{w} \cdot \mathbf{v})\mathbf{v}) = \mathbf{w}.$$

Therefore there exists a vector $\mathbf{u}_0$ such that $\mathbf{u}_0 \wedge \mathbf{v} = \mathbf{w}$. To find the general solution, suppose $\mathbf{u}$ also satisfies the given condition. Then $\mathbf{u}_0 \wedge \mathbf{v} = \mathbf{w} = \mathbf{u} \wedge \mathbf{v} \implies (\mathbf{u} - \mathbf{u}_0) \wedge \mathbf{v} = 0$ so $\mathbf{u} - \mathbf{u}_0$ and $\mathbf{v}$ are parallel. Therefore we have $\mathbf{u} = \mathbf{u}_0 + k\mathbf{v}$ $(k \in \mathbb{R})$.

13. Let $\mathbf{w} = \mathbf{u} \wedge \mathbf{v}$, then

$$\mathbf{w}' = (\mathbf{u} \wedge \mathbf{v})' = \mathbf{u}' \wedge \mathbf{v} + \mathbf{u} \wedge \mathbf{v}' = (a\mathbf{u} + b\mathbf{v}) \wedge \mathbf{v} + \mathbf{u} \wedge (c\mathbf{u} - a\mathbf{v}) = a\mathbf{u} \wedge \mathbf{v} - a\mathbf{u} \wedge \mathbf{v} = \mathbf{0}.$$

Therefore $\mathbf{w}(t) = \mathbf{u}(t) \wedge \mathbf{v}(t)$ is a constant vector.

14. The normal vector of the plane determined by the points $(0,0,0), (1,-2,1), (-1,1,1)$ is parallel to the vector $(1,-2,1) \wedge (-1,1,1) = (-3,-2,-1)$. Now the unit vector we want is perpendicular to both $(2,2,1)$ and $(-3,-2,-1)$, so it is parallel to $(2,2,1) \wedge (-3,-2,-1) = (0,1,2)$. Therefore the answer is $\pm\frac{1}{\sqrt{5}}(0,1,2)$.

1-5. The Local Theory of Curves Parametrized by Arc Length

1. **Note** I will assume $a > 0$. Otherwise, we have to consider the sign of a.

a. $\alpha'(s) = \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right)$, so $|\alpha'(s)| = 1$ and s is the arc length.

b. $\alpha''(s) = \left(-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0\right)$, so the curvature is $k = |\alpha''(s)| = a/c^2$. Moreover, since $\mathbf{n} = \left(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0\right)$, we have

$$\mathbf{b} = \alpha' \wedge \mathbf{n} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_2 \\ -\frac{a}{c}\sin\frac{s}{c} & \frac{a}{c}\cos\frac{s}{c} & \frac{b}{c} \\ -\cos\frac{s}{c} & -\sin\frac{s}{c} & 0 \end{pmatrix} = \frac{1}{c}\left(b\sin\frac{s}{c}, -b\cos\frac{s}{c}, a\right)$$

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and $\mathbf{b}' = \frac{1}{c^2}\left(b\cos\frac{s}{c}, b\sin\frac{s}{c}, 0\right) = -\frac{b}{c^2}\mathbf{n}$. Therefore the torsion is $-b/c^2$.

c. The osculating plane contains $\alpha(s)$ and is normal to the vector $\mathbf{b}$, so it is represented by the equation $\mathbf{b}\cdot(\mathbf{x}-\alpha(s)) = 0$, where $\mathbf{x} = (x,y,z)$, That is,

$$\left(b\sin\frac{s}{c}, -b\cos\frac{s}{c}, a\right)\cdot\left(x - a\cos\frac{s}{c}, y - a\sin\frac{s}{c}, z - b\frac{s}{c}\right) = 0$$

$$\implies \left(b\sin\frac{s}{c}\right)x - \left(b\cos\frac{s}{c}\right)y + a\left(z - b\frac{s}{c}\right) = 0.$$

d. The line containing $\mathbf{n}(s)$ and passing through $\alpha(s)$ is given by

$$\frac{x - a\cos\frac{s}{c}}{-\cos\frac{s}{c}} = \frac{y - a\sin\frac{s}{c}}{-\sin\frac{s}{c}} = \frac{z - b\frac{s}{c}}{0}.$$

Since $\left(0,0,a+b\frac{s}{c}\right)$ satisfies this equation, this line meets the z axis. Also, $\mathbf{n}\cdot\hat{\mathbf{z}} = \mathbf{n}\cdot(0,0,1) = 0$, thus two lines meet under a constant angle $\pi/2$.

e. Since $\mathbf{t} = \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right)$ and $|\mathbf{t}| = |\hat{\mathbf{z}}| = 1$, we have

$$\cos\theta(s) = \mathbf{t}\cdot\hat{\mathbf{z}} = \frac{b}{c} \implies \theta(s) = \arccos\left(\frac{b}{c}\right),$$

which is constant.

2. Since $\alpha' = \mathbf{t}$, $\alpha'' = k\mathbf{n}$, and $\alpha''' = k\mathbf{n}' + k'\mathbf{n} = -k^2\mathbf{t} + k'\mathbf{n} - k\tau\mathbf{b}$, we have $\alpha'\wedge\alpha'' = \mathbf{t}\wedge k\mathbf{n} = k\mathbf{b}$ and $(\alpha'\wedge\alpha'')\cdot\alpha''' = k\mathbf{b}\cdot(-k^2\mathbf{t} + k'\mathbf{n} - k\tau\mathbf{b}) = -k^2\tau$. Therefore

$$\tau(s) = -\frac{\alpha'(s)\wedge\alpha''(s)\cdot\alpha'''(s)}{|k(s)|^2}.$$

3. a. We have $|\mathbf{t}'(s)| = |k(s)\mathbf{n}(s)| = k(s) \neq 0$ by assumption. Therefore $s \mapsto \mathbf{t}(s)$ is a regular parametrized curve.

b. By definition of $\theta(s)$, we get $\mathbf{t}(s) = (\cos\theta(s), \sin\theta(s))$ and $\mathbf{n}(s) = (-\sin\theta(s), \cos\theta(s))$. Since $\mathbf{t}'(s) = (-\theta'(s)\sin\theta(s), \theta'(s)\cos\theta(s)) = \theta'(s)\mathbf{n}(s)$, we have $k = \theta'(s) = d\theta/ds$.

> **Note** The definition of $\theta(s)$ provided in the problem does not guarantee the differentiability of $\theta(s)$, as $\theta(s)$ is not uniquely determined. For example, if $\mathbf{t}(s) = (\cos s, \sin s)$ then both $\theta_1(s) = s$ and $\theta_2(s) = s - 2\pi\lfloor s/(2\pi)\rfloor$ satisfy the definition, yet θ_2 is not differentiable. Therefore, an additional condition, such as continuity, must be imposed.
> We now show that continuity of $\theta(s)$ implies differentiability (of ∞-order). Suppose $\theta(s)$ is continuous. Fix an arbitrary s_0 and consider the case where $\theta(s_0) - 2m\pi \in (-\pi/2, \pi/2)$ for some integer m. Since $\sin\theta(s) = y'(s)$ for $\alpha(s) = (x(s), y(s))$, we can express $\theta(s)$ locally as
> $$\theta(s) = \arcsin(y'(s)) + 2m(s)\pi$$
> for some integer-valued function $m : I \to \mathbb{Z}$. Due to the continuity of $\theta(s)$, The function $m : I \to \mathbb{Z}$ must be locally constant (i.e., $m(s) = m$). Now both $\arcsin(\cdot)$ and $y'(\cdot)$ are C^∞ function, thus $\theta(s)$ is also a C^∞ function at $s = s_0$.
> For cases where $\theta(s_0)$ lies in other intervals, such as $(0, \pi)$, we can proceed similarly using $\arccos(\cdot)$.

4. Since $(\alpha(s) - \mathbf{x}_0) \parallel \mathbf{n}(s)$ for all $s \in I$ with a fixed point $\mathbf{x}_0$, there exists a function $u : I \to \mathbb{R}$ such that $\mathbf{x}_0 = \alpha(s) + u(s)\mathbf{n}(s)$. Differentiate both sides to get

$$\mathbf{0} = \alpha' + u'\mathbf{n} + u\mathbf{n}' = \mathbf{t} + u'\mathbf{n} + u(-k\mathbf{t} - \tau\mathbf{b}) = (1 - uk)\mathbf{t} + u'\mathbf{n} - u\tau\mathbf{b}.$$

Therefore $1 - uk = u' = u\tau = 0$. From $1 - uk = u' = 0$, we have $u(s) = 1/k$. Also, $u\tau = 0$ implies $\tau(s) \equiv 0$, that is, $\alpha(s)$ is contained in a plane. Finally $|\mathbf{x}_0 - \alpha(s)| = |u(s)\mathbf{n}(s)| = 1/k$, so the trace of the curve is contained in a circle.

5. a. Let $\mathbf{x}_0$ be a fixed point, then there exists a function $u : I \to \mathbb{R}$ such that $\mathbf{x}_0 = \alpha(s) + u(s)\mathbf{t}(s)$. Differentiate both sides then
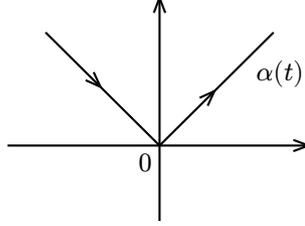
$$\mathbf{0} = \alpha' + u'\mathbf{t} + u\mathbf{t}' = \mathbf{t} + u'\mathbf{t} + uk\mathbf{n} = (1 + u')\mathbf{t} + uk\mathbf{n}.$$

Therefore $1 + u' = uk = 0$. From $u' = -1$, we obtain $u(s) = -s + u_0$ for some constant $u_0 \in \mathbb{R}$. Also, $uk = 0$ implies $k(s) \equiv 0$ (because $k(s) = 0$ for all $s \neq u_0$ and $k(s)$ is continuous at $s = u_0$). Hence $|\alpha''(s)| \equiv 0$ and α is a (segment of a) straight line.

b. No. Consider the curve

$$\alpha(t) = \begin{cases} (t^2, t^2), & (t \geq 0) \\ (-t^2, t^2), & (t < 0) \end{cases}.$$

α has the property that all its tangent lines pass through the fixed point $(0,0)$, but the trace of α is not a straight line.

6. **a.** $|\rho\mathbf{u}| = \sqrt{\rho\mathbf{u} \cdot \rho\mathbf{u}} = \sqrt{\mathbf{u} \cdot \mathbf{u}} = |\mathbf{u}|$, so the norm of a vector is invariant under orthogonal transformations. Also,

$$\cos \angle(\rho\mathbf{u}, \rho\mathbf{v}) = \frac{\rho\mathbf{u} \cdot \rho\mathbf{v}}{|\rho\mathbf{u}||\rho\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \cos \angle(\mathbf{u}, \mathbf{v})$$

and $\angle(\rho\mathbf{u}, \rho\mathbf{v}) = \angle(\mathbf{u}, \mathbf{v})$. Hence the angle between two vectors is also invariant.

b. Let R denotes the matrix representation of ρ, that is, $\rho\mathbf{u} = R\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^3$. Now

$$(\rho\mathbf{u} \wedge \rho\mathbf{v}) \cdot \rho\mathbf{w} = \det(\rho\mathbf{u}, \rho\mathbf{v}, \rho\mathbf{w}) = \det\left(R\mathbf{u} \,\middle|\, R\mathbf{v} \,\middle|\, R\mathbf{w} \right) = \det\left(R\left(\mathbf{u} \,\middle|\, \mathbf{v} \,\middle|\, \mathbf{w} \right) \right)$$

$$= \det(R) \det(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w} = \rho(\mathbf{u} \wedge \mathbf{v}) \cdot \rho\mathbf{w},$$

where $\det(R) = 1$ because ρ is an orthogonal transformation with positive determinant. Since this equation holds for arbitrary $\mathbf{w} \in \mathbb{R}^3$, we have $\rho\mathbf{u} \wedge \rho\mathbf{v} = \rho(\mathbf{u} \wedge \mathbf{v})$.

Note that the condition on the determinant is necessary, otherwise, if $\det(R) = -1$, then $\rho\mathbf{u} \wedge \rho\mathbf{v} = -\rho(\mathbf{u} \wedge \mathbf{v})$ would hold.

c. Let $s(t), k(t), \tau(t)$ be the arc length, the curvature, and the torsion of the curve α. By Exercise 12, we have

$$s'(t) = |\alpha'|, \quad k(t) = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3}, \quad \tau(t) = -\frac{(\alpha' \wedge \alpha'') \cdot \alpha'''}{|\alpha' \wedge \alpha''|^2}.$$

Let $\overline{\alpha}(t) = \rho\alpha(t)$. Then

$$\overline{s}'(t) = |\overline{\alpha}'(t)| = |\rho\alpha'(t)| = |\alpha'(t)| = s'(t),$$

so $\overline{s}(t) = \int_{t_0}^{t} \overline{s}'(t)\,dt = \int_{t_0}^{t} s'(t)\,dt = s(t)$. Also,

$$\overline{k}(t) = \frac{|\rho\alpha' \wedge \rho\alpha''|}{|\rho\alpha'|^3} = \frac{|\rho(\alpha' \wedge \alpha'')|}{|\alpha'|^3} = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3} = k(t).$$

Finally we have

$$\overline{\tau}(t) = -\frac{(\rho\alpha' \wedge \rho\alpha'') \cdot \rho\alpha'''}{|\rho\alpha' \wedge \rho\alpha''|^2} = -\frac{\rho(\alpha' \wedge \alpha'') \cdot \rho\alpha'''}{|\rho(\alpha' \wedge \alpha'')|^2} = -\frac{(\alpha' \wedge \alpha'') \cdot \alpha'''}{|\alpha' \wedge \alpha''|^2} = \tau(t).$$

Therefore the arc length, the curvature, and the torsion are invariant under rigid motions.

7. **a.** We will reparameterize the curve α by arc length. Let $s(t) = \int_{t_0}^{t} |\alpha'(t)|\,dt$ and define the curve $\tilde{\alpha}$ as $\tilde{\alpha}(s) = \alpha(t(s))$, where $t(s)$ is the inverse function of $s(t)$. Note that $\tilde{\alpha}'(s) = \frac{dt}{ds}\frac{d\alpha}{dt} = \frac{|\alpha'(t)|}{|\alpha'(t)|} = 1$, so $\tilde{\alpha}$ is parametrized by arc length. Then we can define $\mathbf{n}(t)$ and $k(t)$ as

$$\mathbf{n}(t) := \tilde{\mathbf{n}}(s(t)), \quad k(t) := \tilde{k}(s(t)),$$

where $\tilde{\mathbf{n}}(s)$ and $\tilde{k}(s)$ are the normal vector and curvature of $\tilde{\alpha}$ at s. Now we have

$$\tilde{\beta}(s) = \tilde{\alpha}(s) + \frac{1}{\tilde{k}(s)}\tilde{\mathbf{n}}(s).$$

Differentiate both sides to get

$$\tilde{\beta}'(s) = \tilde{\alpha}'(s) + \left(\frac{1}{\tilde{k}(s)}\right)' \tilde{\mathbf{n}}(s) + \frac{1}{\tilde{k}(s)}\left(-\tilde{k}(s)\tilde{\mathbf{t}}(s) - \tilde{\tau}(s)\tilde{\mathbf{b}}(s)\right).$$

Let l be the tangent at t of the evolute of α. Since

$$\beta'(t) \cdot \alpha'(t) = |\beta'(t)||\alpha'(t)| \cdot \tilde{\beta}'(s) \cdot \tilde{\alpha}'(s)$$

$$= |\beta'(t)||\alpha'(t)| \cdot \left(\tilde{\alpha}'(s) + \left(\frac{1}{\tilde{k}(s)}\right)' \tilde{\mathbf{n}}(s) - \tilde{\mathbf{t}}(s) - \frac{\tilde{\tau}(s)}{\tilde{k}(s)}\tilde{\mathbf{b}}(s)\right) \cdot \tilde{\alpha}'(s)$$

$$= |\beta'(t)||\alpha'(t)| \cdot (1 + 0 - 1 - 0) = 0,$$

l is normal to α at t. Also, $\alpha(t) = \beta(t) - (1/k(t))\mathbf{n}(t)$, so l passes through the point $\alpha(t)$.

b. Define $s_1 = s(t_1)$, $s_2 = s(t_2)$. Fix t_2, then the intersection point $\mathbf{p}(s_1)$ of the normal lines at t_1 and t_2 is given by

$$\tilde{\alpha}(s_1) + \tilde{\mathbf{n}}(s_1)u(s_1) = \tilde{\alpha}(s_2) + \tilde{\mathbf{n}}(s_2)v(s_1) = \mathbf{p}(s_1),$$

for some function $u, v : I \to \mathbb{R}$. Rewrite this equation by

$$\frac{\tilde{\alpha}(s_2) - \tilde{\alpha}(s_1)}{s_2 - s_1} = \frac{\tilde{\mathbf{n}}(s_1)u(s_1) - \tilde{\mathbf{n}}(s_2)v(s_1)}{s_2 - s_1}$$

and take the inner product of both sides with $\tilde{\mathbf{t}}(s_1)$ to get

$$\tilde{\mathbf{t}}(s_1) \cdot \frac{\tilde{\alpha}(s_2) - \tilde{\alpha}(s_1)}{s_2 - s_1} = \tilde{\mathbf{t}}(s_1) \cdot \frac{\tilde{\mathbf{n}}(s_1)u(s_1) - \tilde{\mathbf{n}}(s_2)v(s_1)}{s_2 - s_1} = \tilde{\mathbf{t}}(s_1) \cdot \frac{-\tilde{\mathbf{n}}(s_2)v(s_1)}{s_2 - s_1}$$

$$= \tilde{\mathbf{n}}(s_2) \cdot \frac{-\tilde{\mathbf{t}}(s_1)v(s_1)}{s_2 - s_1} = \tilde{\mathbf{n}}(s_2) \cdot \frac{\tilde{\mathbf{t}}(s_2) - \tilde{\mathbf{t}}(s_1)}{s_2 - s_1}v(s_1).$$

Therefore

$$v(s_1) = \frac{\tilde{\mathbf{t}}(s_1) \cdot \frac{\tilde{\alpha}(s_2) - \tilde{\alpha}(s_1)}{s_2 - s_1}}{\tilde{\mathbf{n}}(s_2) \cdot \frac{\tilde{\mathbf{t}}(s_2) - \tilde{\mathbf{t}}(s_1)}{s_2 - s_1}},$$

and taking the limit $t_1 \to t_2$ (that is, $s_1 \to s_2$) leads to

$$\lim_{t_1 \to t_2} v(s_1) = \lim_{t_1 \to t_2} \frac{\tilde{\mathbf{t}}(s_1) \cdot \frac{\tilde{\alpha}(s_2) - \tilde{\alpha}(s_1)}{s_2 - s_1}}{\tilde{\mathbf{n}}(s_2) \cdot \frac{\tilde{\mathbf{t}}(s_2) - \tilde{\mathbf{t}}(s_1)}{s_2 - s_1}} = \frac{\tilde{\mathbf{t}}(s_2) \cdot \tilde{\mathbf{t}}(s_2)}{\tilde{\mathbf{n}}(s_2) \cdot \tilde{k}(s_2)\tilde{\mathbf{n}}(s_2)} = \frac{1}{\tilde{k}(s_2)}.$$

Since

$$\lim_{t_1 \to t_2} \mathbf{p}(s_1) = \lim_{t_1 \to t_2} \tilde{\alpha}(s_2) + \tilde{\mathbf{n}}(s_2)v(s_1) = \tilde{\alpha}(s_2) + \frac{1}{\tilde{k}(s_2)}\tilde{\mathbf{n}}(s_2),$$

the intersection points converge to a point on the trace of the evolute of α.

8. **a.** Let $\tilde{\alpha}$ be a reparametrization of α by the arc length. Since $\alpha'(t) = (1, \sinh t)$ and $s(t) = \int_0^t |\alpha'(t)|\, dt = \int_0^t \sqrt{1 + \sinh^2 t}\, dt = \int_0^t \cosh t\, dt = \sinh t$, we have $t(s) = \operatorname{arsinh} s$ as the inverse function of $s(t)$. Therefore

$$\tilde{\alpha}(s) = \alpha(t(s)) = (t(s), \cosh t(s)) = \left(\operatorname{arsinh} s, \sqrt{1 + s^2}\right).$$

The derivatives of $\tilde{\alpha}$ are

$$\tilde{\alpha}'(s) = \left(\frac{1}{\sqrt{1+s^2}}, \frac{s}{\sqrt{1+s^2}}\right), \quad \tilde{\alpha}''(s) = \left(-\frac{s}{(1+s^2)^{3/2}}, \frac{1}{(1+s^2)^{3/2}}\right).$$

Since $\tilde{\mathbf{n}}(s) = \left(-s/\sqrt{1+s^2}, 1/\sqrt{1+s^2}\right)$, we have $\tilde{\alpha}''(s) = \frac{1}{1+s^2}\tilde{\mathbf{n}}(s)$ and $\tilde{k}(s) = \frac{1}{1+s^2}$. Thus

$$k(t) = \tilde{k}(t(s)) = \frac{1}{1 + \sinh^2 t} = \frac{1}{\cosh^2 t}.$$

b. Let $\tilde{\beta}$ be the reparametrization of β by the arc length, where $\beta(t) = \alpha(t) + \frac{1}{k(t)}\mathbf{n}(t)$ is an evolute of α. Then

$$\tilde{\beta}(s) = \tilde{\alpha}(s) + \frac{1}{\tilde{k}(s)}\tilde{\mathbf{n}}(s) = \left(\operatorname{arsinh} s, \sqrt{1+s^2}\right) + (1+s^2)\left(-\frac{s}{\sqrt{1+s^2}}, \frac{1}{\sqrt{1+s^2}}\right)$$

$$= \left(\operatorname{arsinh} s - s\sqrt{1+s^2}, 2\sqrt{1+s^2}\right),$$

so we get

$$\beta(t) = \tilde{\beta}(s(t)) = (t - \sinh t \cosh t, 2\cosh t).$$

9. Since $|\alpha'(s)| = 1$, there exists a differentiable function $\theta : I \to \mathbb{R}$ such that

$$\alpha'(s) = (\cos\theta(s), \sin\theta(s)).$$

Then $\alpha''(s) = \theta'(s)(-\sin\theta(s), \cos\theta(s))$ and $|\alpha''(s)| = \theta'(s)$. By definition we have $\theta'(s) = k(s)$ and $\theta(s) = \int k(s)\,ds + \varphi$ for some constant angle φ. It follows that

$$\alpha(s) = \int \alpha'(s)\,ds = \left(\int \cos\theta(s)\,ds + a, \int \sin\theta(s)\,ds + b\right)$$

for some constant vector (a, b).

10. a. Let $\alpha(t) = (x(t), y(t), z(t))$. Obviously, $x(t) = t$ is differentiable. It remains to show that

$$z(t) = \begin{cases} e^{-1/t^2}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

is differentiable at every $t \in \mathbb{R}$. We assert that there exist a series of polynomials $\{p_n\}_{n=1}^{\infty}$ such that

$$z^{(n)}(x) = \begin{cases} p_n(1/t)e^{-1/t^2}, & t > 0 \\ 0, & t \leq 0 \end{cases} \quad (n \geq 0).$$

For $n = 0$, it is trivial with $p_0(x) = 1$. Now suppose above statement is true for $n = k$. Since $z^{(k)}(t)$ is differentiable (of first order) for every $t \neq 0$, we only have to check for $t = 0$.

$$\lim_{t\to 0^+} \frac{z^{(k)}(t) - z^{(k)}(0)}{t - 0} = \lim_{t\to 0^+} \frac{1}{t}p_k\left(\frac{1}{t}\right)e^{-\frac{1}{t^2}} = \lim_{s\to\infty} sp(s)e^{-s^2} = 0,$$

therefore $z^{(k)}$ is differentiable (of first order) at $t = 0$ and $z^{(k+1)}(0) = 0$. Moreover,

$$z^{(k+1)}(t) = \frac{d}{dt}\left(p_n\left(\frac{1}{t}\right)e^{-\frac{1}{t^2}}\right) = \left(-\frac{1}{t^2}p_n'\left(\frac{1}{t}\right) + \frac{2}{t^3}p_n\left(\frac{1}{t}\right)\right)e^{-\frac{1}{t^2}} \quad (t \geq 0),$$

hence $p_{n+1}(t) = -t^2 p_n'(t) + 2t^3 p_n(t)$.

b. Observe that the derivatives of $\alpha(t)$ are

$$\alpha'(t) = \begin{cases} \left(1, 0, \frac{2}{t^3}e^{-\frac{1}{t^2}}\right), & t > 0 \\ \left(1, \frac{2}{t^3}e^{-\frac{1}{t^2}}, 0\right), & t < 0 \\ (1, 0, 0), & t = 0, \end{cases} \qquad \alpha''(t) = \begin{cases} \left(0, 0, \frac{4-6t^2}{t^6}e^{-\frac{1}{t^2}}\right), & t > 0 \\ \left(0, \frac{4-6t^2}{t^6}e^{-\frac{1}{t^2}}, 0\right), & t < 0 \\ (0, 0, 0), & t = 0. \end{cases}$$

Therefore α is regular for all t. Note that $k(t) = |\alpha'(t) \wedge \alpha''(t)|/|\alpha'(t)|^3$. If $t > 0$, then the curvature is zero if and only if

$$k(t) = 0 \iff |\alpha'(t) \wedge \alpha''(t)| = \left|\left(0, \frac{4-6t^2}{t^6}e^{-\frac{1}{t^3}}, 0\right)\right| = 0 \iff t = \sqrt{\frac{2}{3}}.$$

Similarly, if $t < 0$ then $k(t) = 0$ if and only if $t = -\sqrt{2/3}$. Obviously $k(0) = 0$, since $\alpha''(0) = 0$.

c. If $t > 0$ and $t \to 0$, then the osculating plane is spanned by the vectors $\alpha'(t) = \left(1, 0, \frac{2}{t^3}e^{-\frac{1}{t^2}}\right)$ and $\alpha''(t) = \left(0, 0, \frac{4-6t^2}{t^6}e^{-\frac{1}{t^2}}\right)$, so the plane is $y = 0$. Otherwise, if $t < 0$ and $t \to 0$, then the osculating plane is spanned by the vectors $\alpha'(t) = \left(1, \frac{2}{t^3}e^{-\frac{1}{t^2}}, 0\right)$ and $\alpha''(t) = \left(0, \frac{4-6t^2}{t^6}e^{-\frac{1}{t^2}}, 0\right)$, which implies the plane $z = 0$.

d. Note that $\tau(t) = -((\alpha' \wedge \alpha'') \cdot \alpha''')/|\alpha' \wedge \alpha''|^2$. If $t > 0$, we have $\alpha' \wedge \alpha'' \parallel \hat{\mathbf{y}}$ and $\alpha''' \parallel \hat{\mathbf{z}}$ so $(\alpha' \wedge \alpha'') \cdot \alpha''' = 0$ and $\tau(t) = 0$. Also if $t < 0$, then $\alpha' \wedge \alpha'' \parallel \hat{\mathbf{z}}$ and $\alpha''' \parallel \hat{\mathbf{y}}$ so $(\alpha' \wedge \alpha'') \cdot \alpha''' = 0$ and $\tau(t) = 0$. Finally, if $t = 0$, then $\alpha'' = \alpha''' = 0$ so $\tau(0) = 0$.

11. a. The plane curve is given by
$$\alpha(\theta) = (\rho(\theta)\cos\theta, \rho(\theta)\sin\theta), \quad a \le \theta \le b.$$
Hence $\alpha'(\theta) = (\rho'\cos\theta - \rho\sin\theta, \rho'\sin\theta + \rho\cos\theta)$. The arc length is
$$\int_a^b |\alpha'(\theta)|\, d\theta = \int_a^b \sqrt{(\rho'\cos\theta - \rho\sin\theta)^2 + (\rho'\sin\theta + \rho\cos\theta)^2}\, d\theta = \int_a^b \sqrt{\rho^2 + (\rho')^2}\, d\theta.$$

b. By Exercise 12(d), the curvature of the curve $\alpha(\theta) = (x(\theta), y(\theta))$ is given by
$$k(\theta) = (x'y'' - x''y')/\left((x')^2 + (y')^2\right)^{3/2}.$$

Since
$$\alpha''(\theta) = (\rho''\cos\theta - 2\rho'\sin\theta - \rho\cos\theta, \rho''\sin\theta + 2\rho'\cos\theta - \rho\sin\theta),$$

we have
$$x'y'' - x''y' = (\rho'c - \rho s)(\rho''s + 2\rho'c - \rho s) - (\rho's + \rho c)(\rho''c - 2\rho's - \rho c)$$
$$= 2(\rho')^2 - \rho\rho'' + \rho^2,$$

where $c = \cos\theta$, $s = \sin\theta$. Therefore the curvature is
$$k(\theta) = \frac{x'y'' - x''y'}{\left((x')^2 + (y')^2\right)^{3/2}} = \frac{2(\rho')^2 - \rho\rho'' + \rho^2}{\left((\rho')^2 + \rho^2\right)^{3/2}}.$$

12. a. Note that $s(t) = \int_a^t |\alpha'(t)|\, dt$ and $s'(t) = |\alpha'(t)|$. Hence $\frac{dt}{ds} = 1/(\frac{ds}{dt}) = 1/|\alpha'|$. Also,
$$\frac{d^2t}{ds^2} = \frac{d}{ds}\left(\frac{dt}{ds}\right) = \frac{d}{ds}\left(\frac{1}{|\alpha'(t)|}\right) = \frac{d}{dt}\left(\frac{1}{|\alpha'(t)|}\right) \cdot \frac{dt}{ds} = \frac{-|\alpha'|'}{|\alpha'|^2} \cdot \frac{1}{|\alpha'|} = -\frac{\alpha' \cdot \alpha''}{|\alpha'|^4},$$
where $|\alpha'|' = \left(\sqrt{\alpha' \cdot \alpha'}\right)' = \frac{2\alpha' \cdot \alpha''}{2\sqrt{\alpha' \cdot \alpha'}} = \frac{\alpha' \cdot \alpha''}{|\alpha'|}$.

b. Let $\tilde{k}(s)$ be the curvature of β at $s \in J$, so $\tilde{k}(s) = k(t(s))$. Since $\beta(s) = \alpha(t(s))$, we have $\beta'(s) = \frac{d}{ds}\alpha(t) = \frac{dt}{ds}\frac{d\alpha}{dt} = \alpha'/|\alpha'|$. Moreover,
$$\beta''(s) = \frac{d}{ds}\left(\frac{\alpha'}{|\alpha'|}\right) = \frac{d}{dt}\left(\frac{\alpha'}{|\alpha'|}\right) \cdot \frac{dt}{ds} = \frac{d}{dt}\left(\frac{1}{|\alpha'|} \cdot \alpha'\right) \cdot \frac{dt}{ds}$$
$$= \left(\frac{-|\alpha'|'}{|\alpha'|^2}\alpha' + \frac{1}{|\alpha'|}\alpha''\right) \cdot \frac{1}{|\alpha'|} = \left(-\frac{\alpha' \cdot \alpha''}{|\alpha'|^3}\alpha' + \frac{\alpha' \cdot \alpha'}{|\alpha'|^3}\alpha''\right) \cdot \frac{1}{|\alpha'|}$$
$$= \frac{1}{|\alpha'|^4}\left((\alpha' \cdot \alpha')\alpha'' - (\alpha' \cdot \alpha'')\alpha'\right) = \frac{1}{|\alpha'|^4}\left((\alpha' \wedge \alpha'') \wedge \alpha'\right).$$

Therefore
$$k(t) = \tilde{k}(s) = |\beta''(s)| = \left|\frac{1}{|\alpha'|^4}\left((\alpha' \wedge \alpha'') \wedge \alpha'\right)\right| = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3}$$

because $\alpha' \wedge \alpha''$ and α' are perpendicular.

c. Let $\tilde{\tau}(s)$ be the curvature of β at $s \in J$, so $\tilde{\tau}(s) = \tau(t(s))$. Observe that $\beta''(s) = \tilde{k}(s)\tilde{\mathbf{n}}(s)$ implies $\beta'''(s) = \tilde{k}'\tilde{\mathbf{n}} + \tilde{k}\tilde{\mathbf{n}}' = \tilde{k}'\tilde{\mathbf{n}} + \tilde{k}\left(-\tilde{k}\tilde{\mathbf{n}} - \tilde{\tau}\tilde{\mathbf{b}}\right)$, so $\beta'''(s) \cdot \tilde{\mathbf{b}}(s) = -\tilde{k}(s)\tilde{\tau}(s) = -k(t)\tau(t)$.

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The normal vector $\tilde{\mathbf{n}}(s)$ and binormal vector $\tilde{\mathbf{b}}(s)$ are given by

$$\tilde{\mathbf{n}}(s) = \frac{1}{\tilde{k}(s)}\beta''(s) = \frac{|\alpha'|}{|\alpha' \wedge \alpha''|}\alpha'' - \frac{\alpha' \cdot \alpha''}{|\alpha' \wedge \alpha''|}\frac{\alpha'}{|\alpha'|},$$

$$\tilde{\mathbf{b}}(s) = \tilde{\mathbf{t}}(s) \wedge \tilde{\mathbf{n}}(s) = \frac{\alpha'}{|\alpha'|} \wedge \left(\frac{|\alpha'|}{|\alpha' \wedge \alpha''|}\alpha'' - \frac{\alpha' \cdot \alpha''}{|\alpha' \wedge \alpha''|}\frac{\alpha'}{|\alpha'|} \right) = \frac{\alpha' \wedge \alpha''}{|\alpha' \wedge \alpha''|}.$$

Hence

$$-k(t)\tau(t) = \beta'''(s) \cdot \tilde{\mathbf{b}}(s) = \left(\alpha''' \left(\frac{dt}{ds}\right)^3 + 3\alpha'' \frac{dt}{ds}\frac{d^2t}{ds^2} + \alpha' \frac{d^3t}{ds^3} \right) \cdot \frac{\alpha' \wedge \alpha''}{|\alpha' \wedge \alpha''|} = \frac{\alpha'''}{|\alpha'|^3} \cdot \frac{\alpha' \wedge \alpha''}{|\alpha' \wedge \alpha''|}.$$

Finally we get

$$\tau(t) = -\frac{1}{k(t)}\frac{\alpha'''}{|\alpha'|^3} \cdot \frac{\alpha' \wedge \alpha''}{|\alpha' \wedge \alpha''|} = -\frac{(\alpha' \wedge \alpha'') \cdot \alpha'''}{|\alpha' \wedge \alpha''|^2}.$$

d. Since $\mathbf{t}(t) = \frac{(x',y')}{\sqrt{(x')^2+(y')^2}}$, the normal vector is given by $\mathbf{n}(t) = \frac{(-y',x')}{\sqrt{(x')^2+(y')^2}}$. Note that there is no vector product in $\mathbb{R}^2$, so we can't use the formula $k(t) = |\alpha' \wedge \alpha''|/|\alpha'|^3$. Since we have

$$\beta''(s) = \frac{1}{|\alpha'|^4}((\alpha' \cdot \alpha')\alpha'' - (\alpha' \cdot \alpha'')\alpha') = \frac{(x'',y'')}{(x')^2+(y')^2} - \frac{x'x'' + y'y''}{\left((x')^2+(y')^2\right)^2}(x',y')$$

$$= \frac{x'y'' - x''y'}{\left((x')^2+(y')^2\right)^{3/2}} \cdot \frac{-y',x'}{\sqrt{(x')^2+(y')^2}} = \frac{x'y'' - x''y'}{\left((x')^2+(y')^2\right)^{3/2}}\mathbf{n}(t),$$

the curvature is

$$k(t) = |\beta''(s)| = \frac{x'y'' - x''y'}{\left((x')^2+(y')^2\right)^{3/2}}.$$

13. First suppose that $\alpha(I)$ lies on a sphere. Since k and τ are invariant under translation, we can assume $|\alpha(s)|^2 = C$ for some constant $C > 0$. Differentiate $\alpha(s) \cdot \alpha(s) = C$ three times then we obtain

$$\alpha \cdot \alpha' = 0, \quad \alpha \cdot \alpha'' + \alpha' \cdot \alpha' = \alpha \cdot \alpha'' + 1 = 0, \quad \alpha \cdot \alpha''' + \alpha' \cdot \alpha'' = \alpha \cdot \alpha''' = 0.$$

Since $\alpha'' = k\mathbf{n}$ and $\alpha''' = k'\mathbf{n} + k\mathbf{n}' = k'\mathbf{n} + k(-k\mathbf{t} - \tau\mathbf{b})$, it follows that

$$\alpha \cdot \mathbf{t} = 0, \quad \alpha \cdot k\mathbf{n} = -1, \quad \alpha \cdot (k'\mathbf{n} - k^2\mathbf{t} - k\tau\mathbf{b}) = 0.$$

In particular, the last equation leads to

$$0 = k'(\alpha \cdot \mathbf{n}) - k^2(\alpha \cdot \mathbf{t}) - k\tau(\alpha \cdot \mathbf{b}) = k'\left(-\frac{1}{k}\right) - k^2 \cdot 0 - k\tau(\alpha \cdot \mathbf{b}),$$

hence we get $\alpha \cdot \mathbf{b} = \frac{k'}{k\tau}\left(-\frac{1}{k}\right) = \left(-\frac{k'}{k^2}\right)\frac{1}{\tau} = \left(\frac{1}{k}\right)'\left(\frac{1}{\tau}\right) = R'T$. Thus α satisfies

$$\alpha = (\alpha \cdot \mathbf{t})\mathbf{t} + (\alpha \cdot \mathbf{n})\mathbf{n} + (\alpha \cdot \mathbf{b})\mathbf{b} = -R\mathbf{n} + R'T\mathbf{b}$$

and $|\alpha|^2 = R^2 + (R')^2T^2$ is constant.

Conversely, assume that $R^2 + (R')^2T^2$ is constant, so we have $\left(R^2 + (R')^2T^2\right)' = 2RR' + 2R'T(R'T)' = 0$. We define the parametrized curve $\beta(s) = \alpha(s) + R\mathbf{n} - R'T\mathbf{b}$, then

$$\beta'(s) = \mathbf{t} + R'\mathbf{n} + R(-k\mathbf{t} - \tau\mathbf{b}) - (R'T)'\mathbf{b} - (R'T)\tau\mathbf{n} = -\frac{R}{T}\mathbf{b} - (R'T)'\mathbf{b}$$

$$= -\frac{1}{2R'T}\left(2RR' + 2R'T(R'T)'\right)\mathbf{b} = 0.$$

Therefore $\beta(s)$ is constant, so let $\beta(s) = \mathbf{x}_0$. Accordingly,

$$|\alpha(s) - \mathbf{x}_0| = |R\mathbf{n} - R'T\mathbf{b}| = \sqrt{R^2 + (R')^2T^2} = \text{const.},$$

hence $\alpha(I)$ lies on a sphere.

14. Let $f(t) = |\alpha(t)|^2 = \alpha(t) \cdot \alpha(t)$. Since $f(t)$ is maximum at $t = t_0$, we have $f'(t_0) = 0$ and $f''(t_0) \le 0$, that is,

$$f'(t) = 2\alpha(t) \cdot \alpha'(t) \implies \alpha(t_0) \cdot \alpha'(t_0) = 0,$$

$$f''(t) = 2\alpha'(t) \cdot \alpha'(t) + 2\alpha(t) \cdot \alpha''(t) \implies |\alpha'(t_0)|^2 + \alpha(t_0) \cdot \alpha''(t_0) \le 0.$$

Note that $\alpha \cdot \alpha'' < 0$, so we have $|\alpha \cdot \alpha''| \ge |\alpha'|^2$. Now $\{\alpha(t_0), \alpha'(t_0)\}$ is an orthogonal basis of $\mathbb{R}^2$, so there exists $c_1, c_2 \in \mathbb{R}$ such that $\alpha''(t_0) = c_1 \alpha(t_0) + c_2 \alpha'(t_0)$. Applying inner product with $\alpha(t_0)$ both sides to obtain $\alpha'' \cdot \alpha = c_1 |\alpha|^2$ and $c_1 = (\alpha'' \cdot \alpha)/|\alpha|^2$. By Exercise 12,

$$k(t_0) = \frac{1}{|\alpha'|^4} |(\alpha' \cdot \alpha')\alpha'' - (\alpha' \cdot \alpha'')\alpha'|$$

$$= \frac{1}{|\alpha'|^4} \big||\alpha'|^2(c_1\alpha + c_2\alpha') - c_2|\alpha'|^2\alpha'\big|$$

$$= \frac{|c_1||\alpha|}{|\alpha'|^2} = \frac{|\alpha \cdot \alpha''|}{|\alpha||\alpha'|^2} \ge \frac{|\alpha'|^2}{|\alpha||\alpha'|^2} = \frac{1}{|\alpha(t_0)|}.$$

15. Since $\mathbf{b}' = \tau\mathbf{n}$, the absolute value of the torsion is determined by $|\tau| = |\mathbf{b}'|$, and $\mathbf{n}$ is determined up to a sign. Now $\mathbf{t} = \mathbf{n} \wedge \mathbf{b}$ and the curvature satisfies the relation $\mathbf{t}' = k\mathbf{n}$, so $k > 0$ can be determined.

16. The statement is false. We introduce a counterexample.
For $r, h > 0$ with $r^2 + h^2 = 1$, define a *helix* given by $\alpha(s) = (r\cos s, r\sin s, hs)$. Note that $\alpha'(s) = (-r\sin s, r\cos s, h)$, so $|\alpha'(s)| = \sqrt{r^2 + h^2} = 1$ and α is parametrized by an arc length.
Since $\alpha''(s) = (-r\cos s, -r\sin s) = r(-\cos s, -\sin s, 0)$, we have $k(s) \equiv r$ and $\mathbf{n}(s) = (-\cos s, -\sin s, 0)$. Consequently, $\mathbf{b}(s) = \mathbf{t}(s) \wedge \mathbf{n}(s) = (h\sin s, -h\cos s, r)$ and $\mathbf{b}'(s) = (h\cos s, h\sin s, 0) = -h\mathbf{n}(s)$, thus $\tau(s) \equiv -h$. Observe that $\mathbf{n}(s)$ is the same for every helix with arbitrary $r, h > 0$, but $k(s) = r$, $\tau(s) = -h \neq 0$ is not same. Therefore $k(s)$ and $\tau(s)$ cannot be determined by the normal vector $\mathbf{n}(s)$.

> **Note** In the solution of the book, they said that
> $$\frac{\mathbf{n} \wedge \mathbf{n}' \cdot \mathbf{n}''}{|\mathbf{n}'|^2} = \frac{(k/\tau)'}{(k/\tau)^2 + 1} = (\arctan(k/\tau))' =: a(s),$$

so k/τ can be determined by $\arctan(k/\tau) = \int a(s)\,ds$. However, we have to consider the constant of integration, so $\arctan(k/\tau)$ cannot be determined.
In fact, in the case of family of helix, $(\arctan(k/\tau))' = (\arctan(-r/h))' = 0$ is determined for all helices, but $\arctan(k/\tau) = \arctan(-r/h)$ is not determined.

17. **a.** Let $\mathbf{a}$ be an unit vector of the fixed direction and θ be a fixed angle. By definition, if α is a helix, then $\mathbf{a} \cdot \mathbf{t} = $ const.. Differentiate both sides to get $\mathbf{a} \cdot k\mathbf{n} = 0$, that is, $k(s) = 0$ or $\mathbf{a} \cdot \mathbf{n}(s) = 0$ for every $s \in I$. Note that if $k(s) \equiv 0$ in some interval (a, b), then α is straight line in that interval so $\tau(s) \equiv 0$, which is contradiction. Hence for arbitrary interval (a, b), there is a point $s_0 \in (a, b)$ such that $k(s_0) \neq 0$, that is, $\mathbf{a} \cdot \mathbf{n}(s_0) = 0$. Since $\mathbf{a} \cdot \mathbf{n}(s)$ is continuous, we have $\mathbf{a} \cdot \mathbf{n}(s) = 0$ for all $s \in I$.
Since $(\mathbf{a} \cdot \mathbf{b})' = \mathbf{a} \cdot \tau\mathbf{n} = 0$, we have that $\mathbf{a} \cdot \mathbf{b}$ is constant. Therefore

$$\mathbf{a} \cdot \mathbf{n}' = 0 \implies \mathbf{a} \cdot (-k\mathbf{t} - \tau\mathbf{b}) = 0 \implies \frac{k}{\tau} = -\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{t}} = \text{const.}$$

Conversely, suppose that k/τ is constant. Define $\mathbf{a} = \mathbf{t} - \frac{k}{\tau}\mathbf{b}$. Since

$$\mathbf{a}' = \mathbf{t}' - \frac{k}{\tau}\mathbf{b}' = k\mathbf{n} - \frac{k}{\tau}\tau\mathbf{n} = 0,$$

$\mathbf{a}$ is constant. Because $\mathbf{a} \cdot \mathbf{t} = 1$ is constant, by definition, α is a helix.

b. If α is a helix, then $\mathbf{a} \cdot \mathbf{t} = $ const., so we have $\mathbf{a} \cdot \mathbf{n} = 0$ by the argument of (a). Then the lines containing $\mathbf{n}(s)$ and passing through $\alpha(s)$ are parallel to a fixed plane normal to $\mathbf{a}$.
Conversely, if the last condition holds, then $\mathbf{a} \cdot \mathbf{n} = 0$. So we have $(\mathbf{a} \cdot \mathbf{t})' = \mathbf{a} \cdot k\mathbf{n} = 0$, therefore $\mathbf{a} \cdot \mathbf{t} = $ const. and α is a helix.

c. If α is a helix, $\mathbf{a} \cdot \mathbf{t} = \text{const.}$, so we have $\mathbf{a} \cdot \mathbf{n} = 0$ by the argument of (a). Then we obtain $(\mathbf{a} \cdot \mathbf{b})' = \mathbf{a} \cdot \tau\mathbf{n} = 0$, that is, $\mathbf{a} \cdot \mathbf{b} = \text{const.}$, and the lines containing $\mathbf{b}(s)$ and passing through $\alpha(s)$ make a constant angle $\arccos(\mathbf{a} \cdot \mathbf{b})$ with a fixed direction $\mathbf{a}$.

Conversely, if $\mathbf{a} \cdot \mathbf{b} = \text{const.}$, then $\mathbf{a} \cdot \tau\mathbf{n} = 0$. Since $\tau(s) \neq 0$, we have $\mathbf{a} \cdot \mathbf{n} = 0$. So $(\mathbf{a} \cdot \mathbf{t})' = \mathbf{a} \cdot k\mathbf{n} = 0$ implies that $\mathbf{a} \cdot \mathbf{t} = \text{const.}$ and α is a helix.

18. a. We reparametrize the curve α by an arc length, and denote it by α. (That is, we will assume that α is already parametrized by an arc length.) We can see that this assumption doesn't make any problem. Note that $\overline{\alpha}(s) = \alpha(s) + r(s)\mathbf{n}(s)$, thus $\overline{\alpha}$ may not be parametrized by an arc length. Differentiate both sides with respect to s, then we obtain

$$\frac{d\overline{\alpha}}{ds} = \mathbf{t} + r'\mathbf{n} + r\mathbf{n}' = (1 - rk)\mathbf{t} + r'\mathbf{n} - r\tau\mathbf{b}.$$

By the definition of Bertrand mate, $d\overline{\alpha}/ds$ is tangent to $\mathbf{n}(s)$, so we have $d\overline{\alpha}/ds \cdot \mathbf{n} = r' = 0$ and r is constant.

b. We will denote the arc length, the unit tangent vector, and the normal vector of $\overline{\alpha}$ by $\overline{s}$, $\overline{\mathbf{t}}$ and $\overline{\mathbf{n}}$. First suppose that α is a Bertrand curve. By the definition, we have $\overline{\mathbf{t}} \perp \overline{\mathbf{n}} \parallel \mathbf{n} \perp \mathbf{t}$ for each $s \in I$. Observe that

$$\frac{d(\mathbf{t} \cdot \overline{\mathbf{t}})}{ds} = \mathbf{t} \cdot \frac{d\overline{\mathbf{t}}}{ds} + \frac{d\mathbf{t}}{ds} \cdot \overline{\mathbf{t}} = \mathbf{t} \cdot \frac{d\overline{s}}{ds} \cdot \frac{d\overline{\mathbf{t}}}{d\overline{s}} + \frac{d\mathbf{t}}{ds} \cdot \overline{\mathbf{t}} = \mathbf{t} \cdot \frac{d\overline{s}}{ds} \cdot \overline{k}\overline{\mathbf{n}} + k\mathbf{n} \cdot \overline{\mathbf{t}} = 0,$$

thus $\mathbf{t} \cdot \overline{\mathbf{t}} = \text{const.}$, and $\left|\mathbf{t} \wedge \overline{\mathbf{t}}\right| = \sqrt{1 - (\mathbf{t} \cdot \overline{\mathbf{t}})^2}$ is also constant. Moreover,

$$\mathbf{t} \cdot \overline{\mathbf{t}} = \mathbf{t} \cdot \frac{ds}{d\overline{s}} \cdot \frac{d\overline{\alpha}}{ds} = \mathbf{t} \cdot \frac{ds}{d\overline{s}} \cdot ((1 - rk)\mathbf{t} - r\tau\mathbf{b}) = \frac{ds}{d\overline{s}}(1 - rk),$$

$$\left|\mathbf{t} \wedge \overline{\mathbf{t}}\right| = \left|\mathbf{t} \wedge \left(\frac{ds}{d\overline{s}} \cdot \frac{d\overline{\alpha}}{ds}\right)\right| = \left|\frac{ds}{d\overline{s}}(\mathbf{t} \wedge ((1 - rk)\mathbf{t} - r\tau\mathbf{b}))\right| = \left|\frac{ds}{d\overline{s}}r\tau\right|.$$

So we have $(1 - rk)/(r\tau) = \text{const.} = C$, that is, $rk + Cr\tau = 1$. Letting $A = t$ and $B = Cr$, we obtain $Ak + B\tau = 1$.

Conversely, assume that $Ak + B\tau = 1$ for some constants A, B. Define $\overline{\alpha} = \alpha + r\mathbf{n}$ with $r = A$. We will justify that $\overline{\alpha}$ is Bertrand mate of α. Since

$$\frac{d\overline{\alpha}}{ds} = (1 - rk)\mathbf{t} - r\tau\mathbf{b} = B\tau\mathbf{t} - r\tau\mathbf{b} = \tau(B\mathbf{t} - r\mathbf{b}),$$

the unit vector $\overline{\mathbf{t}}$ of $\overline{\alpha}$ is given by $\overline{\mathbf{t}} = (B\mathbf{t} - r\mathbf{b})/\sqrt{B^2 + r^2}$. It follows that

$$\frac{d\overline{\mathbf{t}}}{ds} = \left((Bk - r\tau)/\sqrt{B^2 + r^2}\right)\mathbf{n},$$

so $\overline{\mathbf{n}}(s) = \pm\mathbf{n}(s)$ and the normal lines of $\overline{\alpha}$ and α at s agree. Thus, α is a Bertrand curve.

c. Assume that α has two distinct Bertrand mates $\overline{\alpha} = \alpha + \overline{r}\mathbf{n}$, $\tilde{\alpha} = \alpha + \tilde{r}\mathbf{n}$. By part (b), we can see that $\alpha + r\mathbf{n}$ is Bertrand mate of α if and only if the relation $rk(s) + B\tau(s) = 1$ holds for some constant B. Hence we obtain $\overline{r}k + \overline{B}\tau = \tilde{r}k + \tilde{B}\tau = 1$ for some constant $\overline{B}, \tilde{B}$. Now for arbitrary $\xi \in \mathbb{R}$, the following relation

$$(\xi\overline{r} + (1 - \xi)\tilde{r})k + \left(\xi\overline{B} + (1 - \xi)\tilde{B}\right)\tau = 1$$

holds. Hence for $r^* = \xi\overline{r} + (1 - \xi)\tilde{r}$, the curve $\alpha^* = \alpha + r^*\mathbf{n}$ is a Bertrand mate of α, so α has infinitely many Bertrand mates.

As $k + \left(\overline{B}/\overline{r}\right)\tau = 1/\overline{r}$ and $k + \left(\tilde{B}/\tilde{r}\right)\tau = 1/\tilde{r}$, it must be $\overline{B}/\overline{r} \neq \tilde{B}/\tilde{r}$. Otherwise, we have $\overline{r} = \tilde{r}$, which is a contradiction. Hence we obtain $k' + \left(\overline{B}/\overline{r}\right)\tau' = k' + \left(\tilde{B}/\tilde{r}\right)\tau' = 0$ and $k' = \tau' = 0$. Using the uniqueness part of the fundamental theorem of the local theory of curves, we can conclude that the circular helix is the only such curve.

2-2. Regular Surfaces; Inverse Images of Regular Values

1. First, we show that the cylinder $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ is a regular surface. Define $f : \mathbb{R}^3 \to \mathbb{R}$ as $f(x, y, z) = x^2 + y^2$. Since $(f_x, f_y, f_z) = (2x, 2y, 0)$, the partial derivatives vanish simultaneously at $(0, 0, z)$ for all $z \in \mathbb{R}$, and the set of critical values is $\{f(0, 0, z) \mid z \in \mathbb{R}\} = \{0\}$. Hence 1 is a regular value of f, and $f^{-1}(1) = S$ is a regular surface by Prop. 2.

We now show that the mapping $\mathbf{x} : U \to S$, $\mathbf{x}(u, v) = (\cos u, \sin u, v)$ where $U = (0, 2\pi) \times \mathbb{R}$ defines a parametrization. Note that $\cos^2 u + \sin^2 u = 1$ so $\mathbf{x}(U) \subset S$. (i) It is easy to see that $\mathbf{x}$ is differentiable. (iii) Since

$$d\mathbf{x_q} = \begin{pmatrix} -\sin u & 0 \\ \cos u & 0 \\ 0 & 1 \end{pmatrix},$$

the differential $d\mathbf{x_q}$ is one-to-one for each $\mathbf{q} \in U$. (ii) Let $(\cos u, \sin u, v) = (x, y, z)$. Since $v = z$ and (x, y) determines u, $\mathbf{x}$ is one-to-one. By Prop. 4, $\mathbf{x}^{-1}$ is continuous.

Note that this parametrization doesn't cover the points $\{(1, 0, v) \mid v \in \mathbb{R}\}$. However, we can define $\mathbf{x}^* : U \to S$, $\mathbf{x}^*(u, v) = (\sin u, \cos u, v)$, and fully cover S by $\mathbf{x}(U) \cup \mathbf{x}^*(U)$.

Note The process of proving that a given mapping $\mathbf{x} : U \to \mathbb{R}^3$ is a parametrization for S is as follows.

- First, justify that $\mathbf{x} : U \to S$ by showing that $\mathbf{x}(U) \subset S$.
- Second, verify the three conditions provided in Def. 1:

  (i) $\mathbf{x}$ is differentiable,
  (ii) $\mathbf{x}$ is a homeomorphism,
  (iii) For each $\mathbf{q} \in U$, $d\mathbf{x_q} : R^2 \to R^3$ is one-to-one.

In the solution above, the notations (i), (ii), and (iii) refer to the proofs of these three specific conditions. The method of proof is as follows:

- If it is easily established by other means that S is a regular surface, we first prove (i) and (iii), and then apply Prop. 4 to conclude (ii).
- Otherwise, conditions (i), (ii), and (iii) must be proven independently.

2. The set $\{(x, y, z) \in \mathbb{R}^3 \mid z = 0, x^2 + y^2 < 1\}$ is a regular surface. It's because $T = \{(x, y) \mid x^2 + y^2 < 1\}$ is open and $f : T \to \mathbb{R}$ defined by $f(x, y) = 0$ is differentiable, so $\{(x, y, f(x, y)) \mid (x, y) \in T\}$ is a regular surface by Prop. 1.

We now show that $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0, x^2 + y^2 \leq 1\}$ is not a regular surface. Let $\mathbf{p} = (1, 0, 0)$, and suppose that S is a regular surface. Then by definition, there exists a neighborhood V of $\mathbf{p}$ in $\mathbb{R}^3$ and a map $\mathbf{x} : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that (i) $\mathbf{x}$ is differentiable, (ii) $\mathbf{x}$ is a homeomorphism, (iii) $d\mathbf{x_q}$ is one-to-one for each $\mathbf{q} \in U$. Since $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$ is onto $V \cap S$, we have $z(u, v) \equiv 0$. Let $\mathbf{q} = \mathbf{x}^{-1}(\mathbf{p})$.

$$\mathbf{F} = \pi \circ \mathbf{x}$$
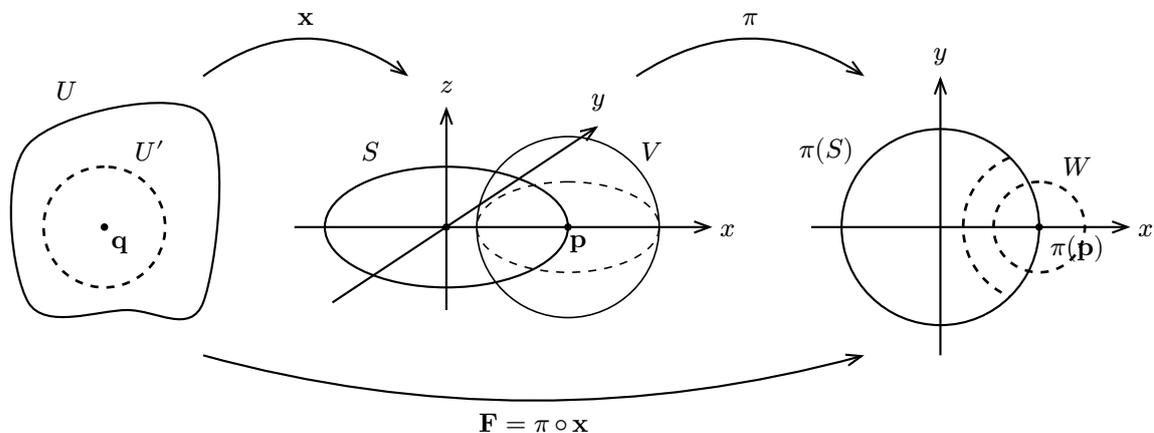
Now define $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ as $\pi(x, y, z) = (x, y)$ and consider the function $\mathbf{F} : \mathbb{R}^2 \to \pi(S)$, $\mathbf{F} = \pi \circ \mathbf{x}$. Since

$$dx_{\mathbf{q}} = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \\ 0 & 0 \end{pmatrix}$$

is one-to-one, we have

$$\left| \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} \right| \neq 0.$$

Therefore $d\mathbf{F_q} = \left| \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} \right| \neq 0$ and we can apply the inverse function theorem, so there exists neighborhood U' of $\mathbf{q}$ in U and neighborhood W of $\mathbf{F(q)} = (1,0)$ in $\mathbb{R}^2$ such that $\mathbf{F}|_{U'} : U' \to W$ has differentiable inverse $(\mathbf{F}|_{U'})^{-1} : W \to U'$.

Let $\mathbf{r} = \mathbf{F(q)} + (\varepsilon, 0)$ for sufficiently small ε so that $\mathbf{r} \in W - \pi(S)$. Then $\mathbf{s} = (\mathbf{F}|_{U'})^{-1}(\mathbf{r}) \in U' \subset U$ and $\mathbf{F(s)} = \mathbf{r} \notin \pi(S)$. However, since $\mathbf{F} : U \to \pi(S)$, it must be $\mathbf{F(s)} \in \pi(S)$, which is a contradiction.

3. If it is a regular surface, by Prop. 3, there exists a representation of the surface in one of three forms $z = f(x,y),\ y = g(x,z),\ x = h(y,z)$ in the neighborhood of $(0,0,0)$. However, projections of the surface over the $xy,\ yz,\ zx$ planes are not one-to-one, so it cannot be the regular surface.
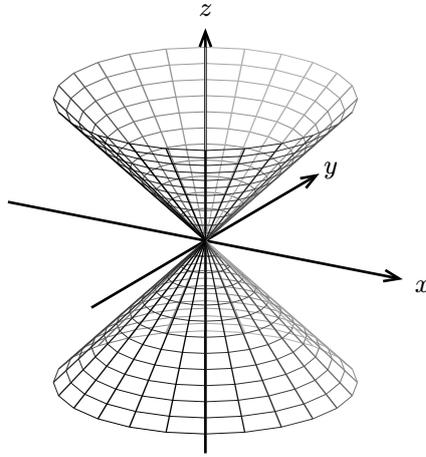
4. Since $df_{\mathbf{q}} = (0,0,2z)$, The critical points of f are the points with $z = 0$ and critical values are $\{f(x,y,0) \mid x,y \in \mathbb{R}\} = \{0\}$. Thus 0 is not a regular value of f. However, $f^{-1}(0)$ is xy-plane, which is a regular surface by Prop. 1 (with $U = \mathbb{R}^2$ and $f(x,y) = 0$).

5. Yes. First observe that P is a regular surface by Prop. 2, because it is given by the preimage of a differentiable function $f(x,y,z) = x - y$ at a regular value 0, that is, $P = f^{-1}(0)$. Now we show that $\mathbf{x}(u,v) = (u+v, u+v, uv)$ is a parametrization of P. Clearly, $\mathbf{x}(U) \subset P$. (i) It is easy to see that $\mathbf{x}$ is differentiable. (iii) Since

$$dx_{\mathbf{q}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ v & u \end{pmatrix}, \qquad \frac{\partial(x,z)}{\partial(u,v)} = u - v \neq 0,$$

the differential $dx_{\mathbf{q}}$ is one-to-one for each $\mathbf{q} \in U$. (ii) Let $(u+v, u+v, uv) = (x,y,z)$. Since u and v are zeros of quadratic equation $t^2 - xt + z = 0$ and $u > v$, we can determine (u,v) by (x,y,z). So $\mathbf{x}$ is one-to-one. By Prop. 4, $\mathbf{x}^{-1}$ is continuous.

6. Let $U \subset \mathbb{R}^2$ be an open set and suppose that $f : U \to \mathbb{R}$ is a differentiable function. Now define the function $h : U \times \mathbb{R} \to \mathbb{R}$ as $h(x,y,z) = f(x,y) - z$. Since $dh_{\mathbf{q}} = (f_x, f_y, -1)$ never vanishes, there is no critical point. Hence every real number is a regular value of h. In particular, 0 is a regular value, so $h^{-1}(0) = \{(x,y,z) \in U \times \mathbb{R} \mid z = f(x,y)\} = \{(x,y,f(x,y)) \mid (x,y) \in U\}$ is a regular surface by Prop. 2.

7. a. Since $df_{\mathbf{q}} = (2(x+y+z-1), 2(x+y+z-1), 2(x+y+z-1))$, the critical points are $\{(x,y,z) \in \mathbb{R}^3 \mid x+y+z = 1\}$. Hence critical values are

$$f(x,y,z)|_{x+y+z=1} = (x+y+z-1)^2|_{x+y+z=1} = 0.$$

b. For $c \neq 0$, c is a regular value of f, so $f^{-1}(c)$ is a regular surface by Prop. 2. If $c = 0$, the surface is given by $x + y + z = 1$, which is plane and also a regular surface.

c. Let $f(x, y, z) = xyz^2$. Since $df_\mathbf{q} = (yz^2, xz^2, 2xyz) = (0, 0, 0)$ implies $x = y = 0$ or $z = 0$, the critical value of f is 0. Now we determine whether $f^{-1}(c)$ is a regular surface. For $c \neq 0$, c is a regular value of f, so $f^{-1}(c)$ is a regular surface by Prop. 2. Otherwise, if $c = 0$, then the surface is given by $f^{-1}(0) = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ or } y = 0 \text{ or } z = 0\}$. Observe that, in the neighborhood of $(0, 0, 0)$, projections of the surface over the xy, yz, zx planes are not one-to-one, so it can not be the regular surface by Prop. 3.

8. Note that two column vectors of $d\mathbf{x}_\mathbf{q}$ are $\partial\mathbf{x}/\partial u$ and $\partial\mathbf{x}/\partial v$. Hence $d\mathbf{x}_\mathbf{q}$ is one-to-one if and only if two column vectors are linearly independent, that is, two vectors are not parallel, so we obtain $\partial\mathbf{x}/\partial u \wedge \partial\mathbf{x}/\partial v \neq 0$.

9. Let $f(x, y, z) = 0$. Then the set $\{(x, y, z) \in \mathbb{R}^3 \mid z = 0 \text{ and } (x, y) \in V\}$ is the same as

$$\{(x, y, f(x, y)) \mid (x, y) \in V\},$$

which is a regular surface by Prop. 1.

10. No. At an intersection point of "8", projections of the surface over the xy, yz, zx planes are not one-to-one. So by Prop. 3, it can not be the regular surface.

11. **a.** Note that the given set $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 - y^2\}$ is a regular surface by Prop. 1 (with $U = \mathbb{R}^2$ and $f(x, y) = x^2 - y^2$.) Now we show that $\mathbf{x}(u, v) = (u + v, u - v, 4uv)$, $(u, v) \in \mathbb{R}^2$ is a parametrization for S. Note that $(u + v)^2 - (u - v)^2 = 4uv$, so we have $\mathbf{x}(U) \subset S$. (i) It is easy to show that $\mathbf{x}$ is differentiable. (iii) Since

$$d\mathbf{x}_\mathbf{q} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 4v & 4u \end{pmatrix}, \qquad \frac{\partial(x, y)}{\partial(u, v)} = -2 \neq 0,$$

the differential $d\mathbf{x}_\mathbf{q}$ is one-to-one for each $\mathbf{q} \in \mathbb{R}^2$. (ii) Let $(\cos u, \sin u, v) = (x, y, z)$. Since $u = \frac{x+y}{2}$ and $v = \frac{x-y}{2}$, $\mathbf{x}$ is one-to-one. By Prop. 4, $\mathbf{x}^{-1}$ is continuous. Therefore $\mathbf{x}$ is a parametrization for S. Note that $\mathbf{x}(\mathbb{R}^2) = S$.

b. Let $U = \{(u, v) \in \mathbb{R}^2 \mid u \neq 0\}$. We want to show that $\mathbf{x}(u, v) = (u \cosh v, u \sinh v, u^2)$ is a parametrization for S. Since $(u \cosh v)^2 - (u \sinh v)^2 = u^2$, we obtain $\mathbf{x} : U \to S$. (i) Obviously $\mathbf{x}$ is differentiable. (iii) Since

$$d\mathbf{x}_\mathbf{q} = \begin{pmatrix} \cosh v & u \sinh v \\ \sinh v & u \cosh v \\ 2u & 0 \end{pmatrix}, \qquad \frac{\partial(x, y)}{\partial(u, v)} = u \neq 0,$$

the differential $d\mathbf{x}_\mathbf{q}$ is one-to-one for each $\mathbf{q} \in U$. (ii) Let $(u \cosh v, u \sinh v, u^2) = (x, y, z)$. By the equation $(\cosh v, \sinh v) = (x/z, y/z)$, we can determine v from (x, y, z). Then $u = x/\cosh v$ can be determined. So $\mathbf{x}$ is one-to-one and $\mathbf{x}^{-1}$ is continuous by Prop. 4. Therefore $\mathbf{x}$ is a parametrization for S. Note that $\mathbf{x}(U) = S|_{z>0}$.
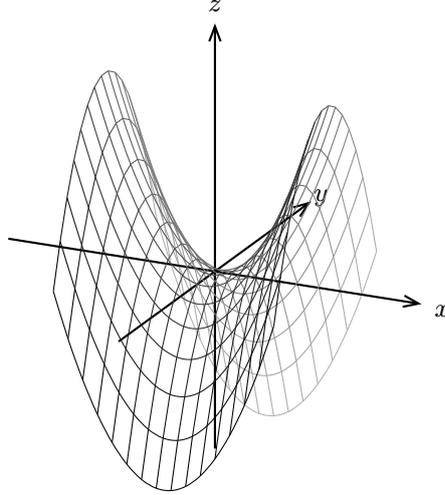
12. First, the ellipsoid $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2/a^2 + y^2/b^2 + z^2/c^2 = 1\}$ is a regular surface by Prop. 2, because it is the preimage of the differentiable function $f(x, y, z) = x^2/a^2 + y^2/b^2 + z^2/c^2$ at the regular value 1.

Now we show that $\mathbf{x}(u, v) = (a \sin u \cos v, b \sin u \sin v, c \cos u)$ is a parametrization for S. Note that

$$\frac{(a \sin u \cos v)^2}{a^2} + \frac{(b \sin u \sin v)^2}{b^2} + \frac{(c \cos u)^2}{c^2} = 1,$$

hence $\mathbf{x}(U) \subset S$. (i) It is easy to see that $\mathbf{x}$ is differentiable. (iii) Since

$$d\mathbf{x_q} = \begin{pmatrix} a \cos u \cos v & -a \sin u \sin v \\ b \cos u \sin v & b \sin u \cos v \\ -c \sin u & 0 \end{pmatrix},$$

we have

$$\frac{1}{a^2 b^2} \left(\frac{\partial(x, y)}{\partial(u, v)} \right)^2 + \frac{1}{a^2 c^2} \left(\frac{\partial(x, z)}{\partial(u, v)} \right)^2 + \frac{1}{b^2 c^2} \left(\frac{\partial(y, z)}{\partial(u, v)} \right)^2$$
$$= \sin^2 u \cos^2 u + \sin^4 u \sin^2 v + \sin^4 u \cos^2 v = \sin^2 u \neq 0,$$

so the differential $d\mathbf{x_q}$ is one-to-one for each $\mathbf{q} \in U$. (ii) Let $(a \sin u \cos v, b \sin u \sin v, c \cos u) = (x, y, z)$. u can be determined by the equation $\cos u = z/c$, and then v is determined by $(\cos v, \sin v) = (x/(a \sin u), y/(b \sin u))$. Hence $\mathbf{x}$ is one-to-one, and $\mathbf{x}^{-1}$ is continuous by Prop. 4. Therefore $\mathbf{x}$ is a parametrization for S.

The curve $u = \text{const.}$ is given by the circle parallel to xy-plane.
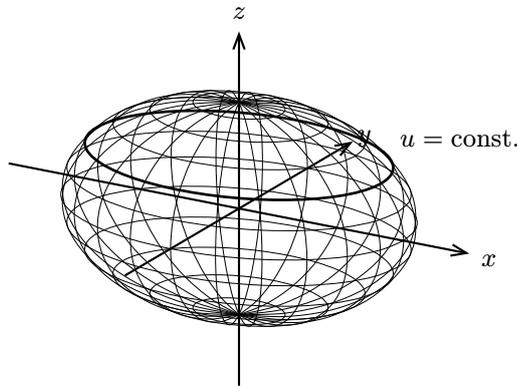
13. Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid -x^2 - y^2 + z^2 = 1\}$. Since S is the preimage of a differentiable function $f(x, y, z) = -x^2 - y^2 + z^2$ at a regular value 0, S is a regular surface. Observe that

$$-(\cos u \sinh v)^2 - (\sin u \sinh v)^2 + (\cosh v)^2 = 1,$$

so we can define $\mathbf{x} : U \to S$ as $\mathbf{x}(u, v) = (\cos u \sinh v, \sin u \sinh v, \cosh v)$ for an open set $U = \{(u, v) \in \mathbb{R}^2 \mid 0 < u < 2\pi, \ v > 0\}$. We will show that $\mathbf{x}$ is a parametrization for the surface S. (i) It is trivial that $\mathbf{x}$ is differentiable. (iii) Since

$$d\mathbf{x_q} = \begin{pmatrix} -\sin u \sinh v & \cos u \cosh v \\ \cos u \sinh v & \sin u \cosh v \\ 0 & \sinh v \end{pmatrix}, \qquad \frac{\partial(x, y)}{\partial(u, v)} = -\sinh v \cosh v \neq 0,$$

the differential $d\mathbf{x_q}$ is one-to-one for each $\mathbf{q} \in U$. (ii) Let $(\cos u \sinh v, \sin u \sinh v, \cosh v) = (x, y, z)$. $v > 0$ can be determined by $\cosh v = z$, then the equation $(\cos u, \sin u) = (x/\sinh v, y/\sinh v)$ determines $u \in (0, 2\pi)$. Hence $\mathbf{x}$ is one-to-one, and $\mathbf{x}^{-1}$ is continuous by Prop. 4. Therefore $\mathbf{x}$ is a parametrization for S. Note that $\mathbf{x}(U)$ only covers the portion of the surface S where $z > 1$. To fully cover S, we define $\mathbf{x}^*(u, v) = (\cos u \sinh v, \sin u \sinh v, -\cosh v)$ for $U^* = \{(u, v) \in \mathbb{R}^2 \mid 0 < u < 2\pi, \ v > 0\}$. Then, for a sufficiently small $\varepsilon > 0$,

$$S = \mathbf{x}(U) \cup \left\{(x, y, z) \in \mathbb{R}^3 \mid z = \sqrt{x^2 + y^2 + 1}, \ x^2 + y^2 < \varepsilon^2\right\}$$

$$\cup \, \mathbf{x}^*(U^*) \cup \left\{(x, y, z) \in \mathbb{R}^3 \mid z = -\sqrt{x^2 + y^2 + 1}, \ x^2 + y^2 < \varepsilon^2\right\}.$$
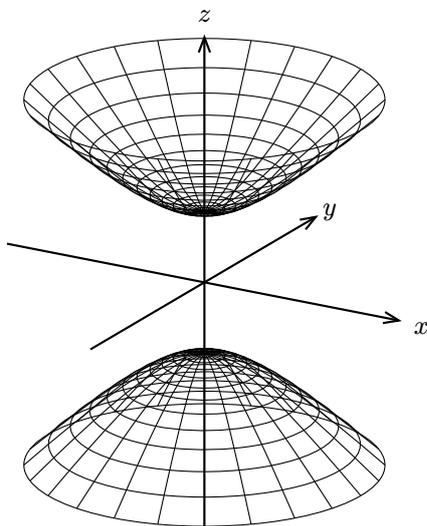
14.
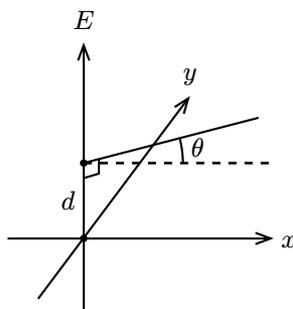
Let t denote the distance from a point to the origin, then the image of the rotating line which has removed the line E is $S = \left\{(t \cos \theta, t \sin \theta, \sin^2(\theta/2)) \mid t > 0, \ \theta \in [0, 2\pi)\right\}$. Hence we define $\mathbf{x} : U \to S$ as

$$\mathbf{x}(t, \theta) = \left(t \cos \theta, t \sin \theta, \sin^2 \frac{\theta}{2}\right), \quad U = \{(t, \theta) \mid t > 0, \ \theta \in (0, 2\pi)\}.$$

Now we will show that $\mathbf{x}$ is a parametrization for S. (i) It is easy to see that $\mathbf{x}$ is differentiable.
(ii) Let $\left(t \cos \theta, t \sin \theta, \sin^2 \frac{\theta}{2}\right) = (x, y, z)$. First, $\mathbf{x}$ has an inverse since $(t \cos \theta, t \sin \theta) = (x, y)$ and the condition $t > 0, \ \theta \in (0, 2\pi)$ completely determines (t, θ). Now we have to verify that t, θ are continuous

functions of (x, y, z). It is easy to see that $t = \sqrt{x^2 + y^2}$ is continuous function of (x, y, z). Observe that if $\theta \neq \pi$, we obtain,

$$\tan \frac{\theta}{2} = \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}} = \frac{\sin \theta}{1 + \cos \theta} = \frac{\frac{y}{t}}{1 + \frac{x}{t}} = \frac{y}{x + t},$$

hence $\theta = 2 \arctan\left(\frac{y}{x+t}\right)$. Thus, if $\theta \neq \pi$, θ is a continuous function of (x, y, z). By the same token, if θ is in a small interval about π, we obtain

$$\theta = 2 \operatorname{arccot}\left(\frac{y}{-x + t}\right).$$

Therefore θ is a continuous function of (x, y, z). This shows that $\mathbf{x}^{-1}$ is continuous. (iii) Since

$$d\mathbf{x_q} = \begin{pmatrix} \cos \theta & -t \sin \theta \\ \sin \theta & t \cos \theta \\ 0 & \frac{1}{2} \sin \theta \end{pmatrix}, \qquad \frac{\partial(x, y)}{\partial(t, \theta)} = t \neq 0,$$

the differential $d\mathbf{x_q}$ is one-to-one for each $\mathbf{q} \in U$. Therefore $\mathbf{x}$ is a parametrization for S.
Note that this parametrization doesn't cover the points $\{(t, 0, 0) \mid t > 0\} \subset S$. To resolve this, we can define

$$\mathbf{x}^*(t, \theta) = \left(t \cos \theta, t \sin \theta, \sin^2 \frac{\theta}{2}\right), \quad U^* = \{(t, \theta) \mid t > 0, \ \theta \in (-\varepsilon, \varepsilon)\}$$

for sufficiently small $\varepsilon > 0$, and fully cover the S by $\mathbf{x}(U) \cup \mathbf{x}^*(U^*)$. Hence S is a regular surface.
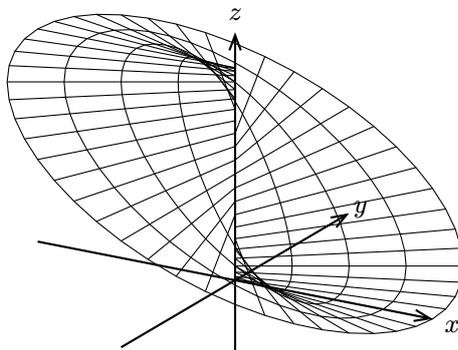
Let's consider the case when $d = \sin(\theta/2)$. Note that $\sin(\theta/2)$ is a function of period 4π. If we denote t the distance from a point to the origin, then the image of the rotating line which has removed the line E is $S = \{(t \cos \theta, t \sin \theta, \sin(\theta/2)) \mid t > 0, \ \theta \in [0, 4\pi)\}$.
Observe that in the neighborhood of $(t, 0, 0), t > 0$, the projections of S over the xy, yz, zx planes are not one-to-one. Hence the line $T = \{(t, 0, 0) \mid t > 0\}$ would need to be excluded from S.
Now define the functions

$$\mathbf{x}(t, \theta) = \left(t \cos \theta, t \sin \theta, \sin \frac{\theta}{2}\right), \quad U = \{(t, \theta) \mid t > 0, \ \theta \in (0, 2\pi)\},$$

$$\mathbf{x}^*(t, \theta) = \left(t \cos \theta, t \sin \theta, \sin \frac{\theta}{2}\right), \quad U^* = \{(t, \theta) \mid t > 0, \ \theta \in (2\pi, 4\pi)\},$$

Then we can show that $\mathbf{x}$ and $\mathbf{x}^*$ is a parametrization for $S - T$ similar to the case $d = \sin^2(\theta/2)$. Finally, $S - T = \mathbf{x}(U) \cup \mathbf{x}^*(U^*)$ is a regular surface.
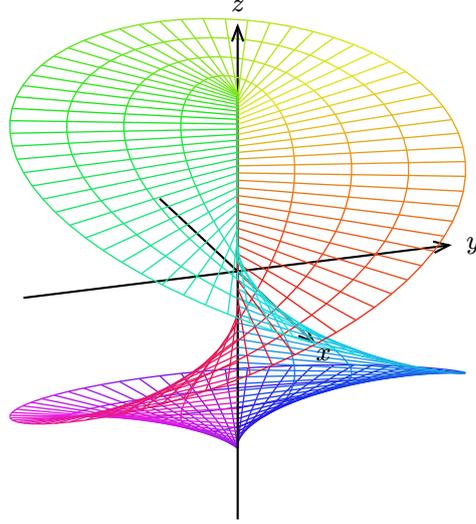
15. We can assume the velocity of the points is 1 and let $\mathbf{p}(t) = (0, 0, t)$ and $\mathbf{q}(t) = (a, t, 0)$. Hence the line passing through $\mathbf{p}(t)$ and $\mathbf{q}(t)$ is given by $\frac{x-0}{a-0} = \frac{y-0}{t-0} = \frac{z-t}{0-t}$, where the convention is made that if a denominator is zero, the corresponding numerator is also zero. We denote

$$T = \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, \frac{x}{a} = \frac{y}{t} = \frac{z-t}{-t} = s \text{ for } t, s \in \mathbb{R} \right\},$$

$$S = \{ (x, y, z) \in \mathbb{R}^3 \mid y(x - a) + zx = 0 \}$$

and prove that $T = S$.

First suppose that $(x, y, z) \in T$. Then $(x, y, z) = (as, ts, t - ts)$, so we have $y(x - a) + zx = ts(as - a) + (t - ts)as = 0$, that is, $(x, y, z) \in S$. Conversely, choose arbitrary $(x, y, z) \in S$ and let $s = \frac{x}{a}$. Now we have $x = as$ and $a \neq 0$, thus

$$y(x - a) + zx = 0 \implies y(as - a) + zas = 0 \implies y(s - 1) + zs = 0 \implies \frac{y}{s} = \frac{z}{1 - s}.$$

Letting $\frac{y}{s} = \frac{z}{1-s} = t$ gives the desired relation $\frac{x}{a} = \frac{y}{t} = \frac{z-t}{t} = s$. Hence $T = S$.
Now we prove that this set S is a regular surface. Define $f : \mathbb{R}^3 \to \mathbb{R}$ as $f(x, y, z) = y(x - a) + zx$. Since $df_{\mathbf{q}} = (y + z, x - a, x) = 0$ implies $x = 0 \wedge x = a$, which is impossible, so every real number is a regular value of f. In particular, 0 is a regular value, thus by Prop. 2, $S = f^{-1}(0)$ is a regular surface.

16. a. Since three points $N = (0, 0, 2)$, (x, y, z), $(u, v, 0)$ are contained in one straight line, we have $\frac{x}{u} = \frac{y}{v} = \frac{z-2}{0-2} = t$. Then put $(x, y, z) = (ut, vt, 2 - 2t)$ into $x^2 + y^2 + (z - 1)^2 = 1$ to obtain

$$u^2t^2 + v^2t^2 + (1 - 2t)^2 = 1 \implies (u^2 + v^2 + 4)t^2 = 4t \implies t = \frac{4}{u^2 + v^2 + 4}.$$

Therefore we have

$$\pi^{-1}(u, v) = (x, y, z) = \left(\frac{4u}{u^2 + v^2 + 4}, \frac{4v}{u^2 + v^2 + 4}, \frac{2(u^2 + v^2)}{u^2 + v^2 + 4} \right).$$

b. First we show that $\mathbf{x} = \pi^{-1} : \mathbb{R}^2 \to S$ is a parametrization for the given sphere. (i) It is easy to see that $\mathbf{x}$ is differentiable. (iii) Since

$$d\mathbf{x}_{\mathbf{q}} = \begin{pmatrix} \frac{-4u^2 + 4v^2 + 16}{(u^2+v^2+4)^2} & \frac{-8uv}{(u^2+v^2+4)^2} \\ \frac{-8uv}{(u^2+v^2+4)^2} & \frac{4u^2 - 4v^2 + 16}{(u^2+v^2+4)^2} \\ \frac{16u}{(u^2+v^2+4)^2} & \frac{16v}{(u^2+v^2+4)^2} \end{pmatrix}, \qquad \frac{\partial(x, z)}{\partial(u, v)} = \frac{64v}{(u^2 + v^2 + 4)^3} \neq 0,$$

the differential $d\mathbf{x}_{\mathbf{q}}$ is one-to-one for each $\mathbf{q} \in U$. (ii) Let $\left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{2(u^2+v^2)}{u^2+v^2+4} \right) = (x, y, z)$. Since $u^2 + v^2 = \frac{4z}{2-z}$ and $(u, v) = \frac{u^2+v^2+4}{4}(x, y)$, (u, v) is determined by (x, y, z). By Prop. 4, $\mathbf{x}^{-1}$ is continuous. Therefore $\mathbf{x}$ is a parametrization for the sphere $\mathbb{S}^2$.

Note that $\mathbf{x}(\mathbb{R}^2) = \mathbb{S}^2 - \{N\}$, so this parametrization doesn't cover the point N. To resolve this issue, define new function $\mathbf{x}^* : \mathbb{R}^2 \to S$ as follows. Consider the another stereographic projection $\pi^* : \mathbb{S}^2 - \{\mathbf{0}\} \to \mathbb{R}^2$ which carries a point $\mathbf{p} = (x, y, z) \in \mathbb{S}^2 - \{\mathbf{0}\}$ onto the intersection of the plane $\{(u, v, 2) \mid (u, v) \in \mathbb{R}^2\}$ with the straight line which connects $\mathbf{0}$ to $\mathbf{p}$. Let $(u, v) = \pi^*(x, y, z)$.

Since three points $\mathbf{0} = (0, 0, 0)$, (x, y, z), $(u, v, 2)$ lie on the same straight line, we obtain $\frac{x}{u} = \frac{y}{v} = \frac{z}{2} = t$. Then put $(x, y, z) = (ut, vt, 2t)$ into $x^2 + y^2 + (z - 1)^2 = 1$ to get

$$u^2 t^2 + v^2 t^2 + (2t - 1)^2 = 1 \implies (u^2 + v^2 + 4)t^2 = 4t \implies t = \frac{4}{u^2 + v^2 + 4}.$$
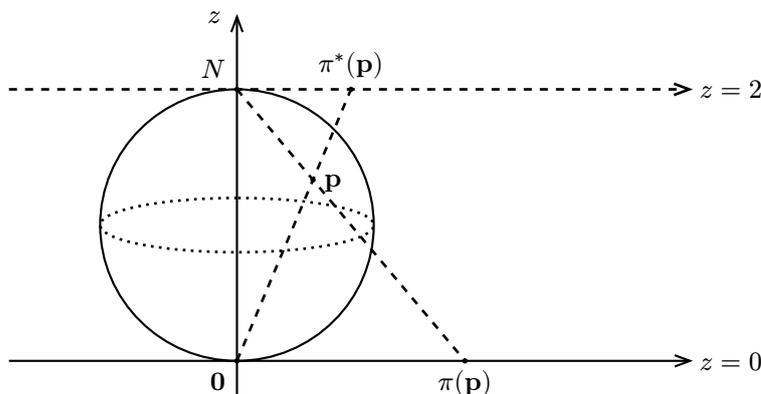
Therefore we have

$$(\pi^*)^{-1}(u, v) = (x, y, z) = \left(\frac{4u}{u^2 + v^2 + 4}, \frac{4v}{u^2 + v^2 + 4}, \frac{8}{u^2 + v^2 + 4} \right).$$

Similar to the argument above, we can show that $\mathbf{x}^* = (\pi^*)^{-1} : \mathbb{R}^2 \to \mathbb{S}^2 - \{\mathbf{0}\}$ is a parametrization for $\mathbb{S}^2$. Finally we have

$$\mathbf{x}(\mathbb{R}^2) \cup \mathbf{x}^*(\mathbb{R}^2) = (\mathbb{S}^2 - \{N\}) \cup (\mathbb{S}^2 - \{\mathbf{0}\}) = \mathbb{S}^2,$$

so it is possible to cover the sphere with two coordinate neighborhoods using stereographic projection.

17. A regular curve in $\mathbb{R}^3$ is a subset $C \subset \mathbb{R}^3$ with following property: for each point $\mathbf{p} \in C$, there is a neighborhood V of $\mathbf{p}$ in $\mathbb{R}^3$ and a differentiable homeomorphism $\alpha : I \to V \cap C$ of an open interval $I \subset \mathbb{R}$ onto $V \cap C$ such that the differential $d\alpha_t$ is one-to-one for each $t \in I$.

a. Suppose $\mathbf{p} = (x_0, y_0)$ be a point of $f^{-1}(a)$. Since a is a regular value of f, it is possible to assume, by renaming the axes if necessary, that $f_y \neq 0$ at $\mathbf{p}$. We define a mapping $\mathbf{F} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\mathbf{F}(x, y) = (x, f(x, y)),$$

and we indicate by (u, v) the coordinates of a point in $\mathbb{R}^2$ where $\mathbf{F}$ takes its values. The differential of $\mathbf{F}$ at $\mathbf{p}$ is given by

$$d\mathbf{F}_{\mathbf{p}} = \begin{pmatrix} 1 & 0 \\ f_x & f_y \end{pmatrix},$$

whence

$$\det(d\mathbf{F}_{\mathbf{p}}) = f_y \neq 0.$$

We can therefore apply the inverse function theorem, which guarantees the existence of neighborhoods V of $\mathbf{p}$ and W of $\mathbf{F}(\mathbf{p})$ such that $\mathbf{F} : V \to W$ is invertible and the inverse $\mathbf{F}^{-1} : W \to V$ is differentiable. It follows that the coordinate functions of $\mathbf{F}^{-1}$, i.e., the functions

$$x = u, \quad y = g(u, v), \quad (u, v) \in W$$

are differentiable. In particular, $y = g(u, a) = h(x)$ is a differentiable function defined in the projection of V onto the x axis. Since

$$\mathbf{F}(f^{-1}(a) \cap V) = W \cap \{(u, v) \mid v = a\},$$

we conclude that the graph of h is $f^{-1}(a) \cap V$. By a similar argument of Prop. 1, $f^{-1}(a) \cap V$ is a coordinate neighborhood of $\mathbf{p}$. Therefore, every $\mathbf{p} \in f^{-1}(a)$ can be covered by a coordinate neighborhood, and so $f^{-1}(a)$ is a regular curve.

Now consider the function $f(x,y) = x^2 - y^2 - 1$. Since $df_{\mathbf{q}} = (2x, -2y) = (0,0)$ implies $(x,y) = (0,0)$, the critical value of f is $f(0,0) = -1$. Hence $f^{-1}(0) = \{(x,y) \in \mathbb{R}^2 \mid x^2 - y^2 = 1\}$ is a regular curve which is not connected.

b. Suppose $\mathbf{p} = (x_0, y_0, z_0)$ be a point of $\mathbf{F}^{-1}(\mathbf{a})$. Since $\mathbf{a} = (a,b)$ is a regular value of $\mathbf{F}$,

$$d\mathbf{F}_{\mathbf{p}} = \begin{pmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \end{pmatrix}$$

is surjective, and at least one of the determinants

$$\left| \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \right|, \quad \left| \begin{pmatrix} f_x & f_z \\ g_x & g_z \end{pmatrix} \right|, \quad \left| \begin{pmatrix} f_y & f_z \\ g_y & g_z \end{pmatrix} \right|$$

is nonzero. Hence it is possible to assume, by renaming the axes if necessary, that $\left| \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \right| \neq 0$ at $\mathbf{p}$. We define a mapping $\mathbf{G} : U \subset \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\mathbf{G}(x,y,z) = (x, f(x,y,z), g(x,y,z)),$$

and we indicate by (u,v,t) the coordinates of a point in $\mathbb{R}^3$ where $\mathbf{G}$ takes its values. The differential of $\mathbf{G}$ at $\mathbf{p}$ is given by

$$d\mathbf{G}_{\mathbf{p}} = \begin{pmatrix} 1 & 0 & 0 \\ f_x & f_y & f_y \\ g_z & g_y & g_z \end{pmatrix},$$

whence

$$\det(d\mathbf{G}_{\mathbf{p}}) = 1 \cdot \left| \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \right| \neq 0.$$

We can therefore apply the inverse function theorem, which guarantees the existence of neighborhoods V of $\mathbf{p}$ and W of $\mathbf{G}(\mathbf{p})$ such that $\mathbf{G} : V \to W$ is invertible and the inverse $\mathbf{G}^{-1} : W \to V$ is differentiable. It follows that the coordinate functions of $\mathbf{G}^{-1}$, i.e., the functions

$$x = u, \quad y = i(u,v,t), \quad z = j(u,v,t), \quad (u,v,t) \in W$$

are differentiable. In particular, $y = i(u,a,b) = \imath(x)$ and $z = j(u,a,b) = \jmath(x)$ is a differentiable function defined in the projection of V onto the x axis. Since

$$\mathbf{G}(\mathbf{F}^{-1}(\mathbf{a}) \cap V) = W \cap \{(u,v,t) \mid v = a, \ t = b\},$$

we conclude that the graph of $(\imath, \jmath)$, that is

$$\{(x, \imath(x), \jmath(x)) \mid (x,y,z) \in V \text{ for some } y, z\},$$

is equal to $\mathbf{F}^{-1}(\mathbf{a}) \cap V$. By a similar argument of Prop. 1, $\mathbf{F}^{-1}(\mathbf{a}) \cap V$ is a coordinate neighborhood of $\mathbf{p}$. Therefore, every $\mathbf{p} \in \mathbf{F}^{-1}(\mathbf{a})$ can be covered by a coordinate neighborhood, and so $\mathbf{F}^{-1}(\mathbf{a})$ is a regualr curve.

Now, using this proposition, we can show that the set defined by the intersection of two surfaces is a curve. If two surfaces are given by $f(x,y,z) = a$ and $g(x,y,z) = b$ respectively, the intersection is given by $\mathbf{F}^{-1}(\mathbf{a})$, where $\mathbf{F}(x,y,z) = (f(x,y,z), g(x,y,z))$ and $\mathbf{a} = (a,b)$. Provided that $\mathbf{a}$ is a regular value of $\mathbf{F}$, the proposition just proved implies that this set is a regular curve.

c. We extend Prop. 3 for plane curves.

Proposition 3 for plane curves. Let $C \subset \mathbb{R}^2$ be a regular curve and $\mathbf{p} \in C$. Then there exists a neighborhood V of $\mathbf{p}$ in C such that V is a graph of a differentiable function which has one of the following two forms: $y = f(x)$, $x = g(y)$.

The proof is as follows. Let $\alpha : I \subset \mathbb{R} \to C$ be a parametrization of C in $\mathbf{p}$, and write $\alpha(t) = (x(t), y(t))$, $t \in I$. By the definition of regular curve, one of the derivatives x_t, y_t is not zero at $\alpha^{-1}(\mathbf{p}) = q$. Suppose first that $x_t \neq 0$, and consider the map $\pi \circ \alpha : I \to \mathbb{R}$, where π is the projection $\pi(x, y) = x$. Then $\pi \circ \alpha(t) = x(t)$, and since $(dx/dt)(q) \neq 0$, we can apply the inverse function theorem to guarantee the existence of neighborhoods V_1 of q, V_2 of $\pi \circ \alpha(q)$ such that $\pi \circ \alpha$ maps V_1 diffeomorphically onto V_2. It follows that π restricted to $\alpha(V_1) = V$ is one-to-one and that there is a differentiable inverse $(\pi \circ \alpha)^{-1} : V_2 \to V_1$. Observe that, since α is a homeomorphism, V is a neighborhood of $\mathbf{p}$ in C. Now, if we compose the map $(\pi \circ \alpha)^{-1} : x \mapsto t(x)$ with the function $t \mapsto y(t)$, we find that V is the graph of the differentiable function $y = y(t) = y(t(x)) = f(x)$, and this settles the first case. The remaining case can be treated in the same way, yielding $x = g(y)$.

Now we get back to the exercise. If $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 = y^3\}$ is a regular curve, then by the proposition above, in a neighborhood of $(0, 0) \in C$, the graph of a differentiable function having one of two forms: $y = f(x)$, $x = g(y)$. The second form can be discarded by the simple fact that the projections of C over the y axis is not one-to-one. The first form would have to agree, in a neighborhood of $(0, 0, 0)$, with $y = x^{2/3}$. Since $y = x^{2/3}$ is not differentiable at 0, this is impossible.
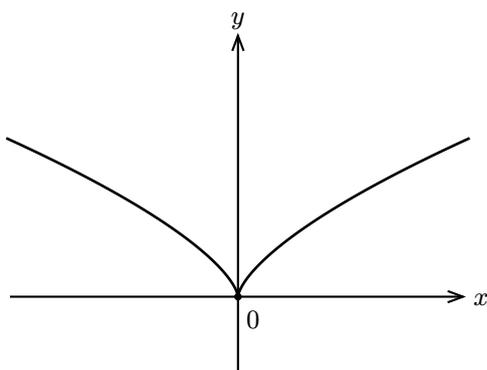
18. Define $\mathbf{E} : \mathbb{R}^3 \to \mathbb{R}^3$ as $\mathbf{E}(x, y, z) = (f(x, y, z), g(x, y, z), h(x, y, z))$. Since

$$\det(d\mathbf{E}_{\mathbf{p}}) = \frac{\partial(f, g, h)}{\partial(x, y, z)}(\mathbf{p}) \neq 0,$$

by the inverse function theorem, there exists a neighborhood V of $\mathbf{p}$ and W of $\mathbf{E}(p)$ such that $\mathbf{E}|_V : V \to W$ is invertible and $(\mathbf{E}|_V)^{-1} : W \to V$ is differentiable. Now denote $\mathbf{F} = (\mathbf{E}|_V)^{-1}$ then we can write

$$(x, y, z) = \mathbf{F}(u, v, w),$$

so three families (x, y, z) in a neighborhood of (x_0, y_0, z_0) can be described by a mapping $\mathbf{F}$.
If $f(x, y, z) = x^2 + y^2 + z^2 = u, g(x, y, z) = y/x = v, h(x, y, z) = (x^2 + y^2)/z^2 = w$, then we have

$$x^2 = \frac{uw}{(v^2 + 1)(w + 1)}, \quad y^2 = \frac{uv^2w}{(v^2 + 1)(w + 1)}, \quad z^2 = \frac{u}{w + 1}$$

$$\implies (x, y, z) = \left(\pm\sqrt{\frac{uw}{(v^2 + 1)(w + 1)}}, \ \pm v\sqrt{\frac{uw}{(v^2 + 1)(w + 1)}}, \ \pm\sqrt{\frac{u}{w + 1}} \right).$$

Note that there are four possibilities for choosing the signs, and that is completely determined by (x_0, y_0, z_0). To see this, let $\mu_1 = \text{sgn}(x_0)$ and $\mu_2 = \text{sgn}(z_0)$, then

$$\mathbf{F}(u, v, w) = \left(\mu_1\sqrt{\frac{uw}{(v^2 + 1)(w + 1)}}, \ \mu_1 v\sqrt{\frac{uw}{(v^2 + 1)(w + 1)}}, \ \mu_2\sqrt{\frac{u}{w + 1}} \right).$$

19. **a.** No, C is not a regular curve. If C is a regular curve, then by Prop. 3 for plane curves, in a neighborhood of $\alpha(-2) = (0, 0) \in C$, it would be a graph of a differentiable function having the form $y = f(x)$ or $x = g(y)$. However, given neighborhood V of $(0, 0)$, we can choose sufficiently small $\varepsilon > 0$ so that $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < \varepsilon^2\} \subset N$ and large natural number $n > 1/(2\pi r)$, then we have

$$\alpha(-2) = (0,0), \quad \alpha\left(-2 - \frac{\varepsilon}{2}\right) = \left(0, \frac{\varepsilon}{2}\right), \quad \alpha\left(-\frac{1}{2n\pi}\right) = \left(\frac{1}{2n\pi}, 0\right),$$

which implies that the projections of C over the x and y axis are not one-to-one. Hence C is not a regular curve.

Note In this book, we distinguish between two types of regular curves:

1. Regular parametrized curves; differentiable map $\alpha : I \to \mathbb{R}^3$ such that $\alpha'(t) \neq \mathbf{0}$ for all $t \in I$.
2. Regular curves; subset $C \subset \mathbb{R}^3$ with following property: for each point $\mathbf{p} \in C$, there is a neighborhood V of $\mathbf{p}$ in $\mathbb{R}^3$ and a differentiable homeomorphism $\alpha : I \subset \mathbb{R} \to V \cap C$ such that the differential $d\alpha_t$ is one-to-one for each $t \in I$.

Note that the second definition is stricter; it requires $V \cap C$ to be homeomorphic to an open interval, which effectively forbids self-intersections. In the context of this exercise, the curve C is a regular parametrized curve since $\alpha'(t) \neq \mathbf{0}$ for all $t \in (-3, 0)$, but it fails to be a regular curve.

b. No. Similar to argument above, we can see that in a neighborhood of $(0,0,0)$, the projections of S over the xy, yz, xz planes are not one-to-one, so it cannot be a regular surface.

2-3. Change of Parameters; Differentiable Functions on Surface

1. Define a mapping $\mathbf{B} : \mathbb{R}^3 \to \mathbb{R}^3$ as $\mathbf{B}(x, y, z) = (-x, -y, -z)$, then $\mathbf{B}$ is differentiable. For arbitrary $\mathbf{p} \in \mathbb{S}^2$, there is a parametrization $\mathbf{x} : U \to \mathbb{S}^2$ with open set $U \subset \mathbb{R}^2$ such that $\mathbf{p} \in \mathbf{x}(U)$. Observe that $\mathbf{B} \circ \mathbf{x}$ is differentiable at any point of U. Since $\mathbf{B}|_{\mathbb{S}^2} = \mathbf{A}$ and $\mathbf{x}(U) \subset \mathbb{S}^2$, we have $\mathbf{A} \circ \mathbf{x} = \mathbf{B} \circ \mathbf{x}$ is differentiable at $\mathbf{x}^{-1}(\mathbf{p})$. Hence $\mathbf{A}$ is differentiable in $\mathbb{S}^2$.

Now $\mathbf{A}^2 = \mathrm{id}_{\mathbb{S}^2}$, so we can see that $\mathbf{A}^{-1} = \mathbf{A}$ is also differentiable, that is, $\mathbf{A}$ is a diffeomorphism.

Note We can generalize this argument. Let S_1 and S_2 be regular surfaces. Assume $S_1 \subset V \subset \mathbb{R}^3$, where V is an open set of $\mathbb{R}^3$, and that $\varphi : V \to \mathbb{R}^3$ is a differentiable map such that $\varphi(S_1) \subset S_2$. Then the restriction $\varphi|_{S_1} : S_1 \to S_2$ is a differentiable map in S_1.

The proof is as follows. If we choose arbitrary $\mathbf{p} \in S_1$ and parametrization $\mathbf{x}_1 : U_1 \to S_1$ and $\mathbf{x}_2 : U_2 \to S_2$ with $\mathbf{p} \in \mathbf{x}_1(U_1)$, $\varphi(\mathbf{x}_1(U_1)) \subset \mathbf{x}_2(U_2)$, we have that the map $\mathbf{x}_2^{-1} \circ \varphi \circ \mathbf{x}_1 : U_1 \to U_2$ is differentiable at $\mathbf{x}_1^{-1}(\mathbf{p})$. (see Example 3.)

2. Note that $\pi : S \to \mathbb{R}^3$ is defined as $\pi(x, y, z) = (x, y, 0)$. Since the mapping $\pi^* : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\pi^*(x, y, z) = (x, y, 0)$ is differentiable and $\pi^*(\mathbb{R}^3) \subset \mathbb{R}^2 \times \{0\}$, so $\pi = \pi^*|_S$ is a differentiable map in S.

3. Denote the paraboloid and the plane by $S_1 = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}$ and $S_2 = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$. We want to show that $\pi : S_1 \to S_2$ defined as $\pi(x, y, z) = (x, y, 0)$ is a diffeomorphism. Since $\pi^* : \mathbb{R}^3 \to \mathbb{R}^3$, $\pi^*(x, y, z) = (x, y, 0)$, the extension of π, is differentiable, and $\pi^*(S_1) \subset S_2$, we have that $\pi = \pi^*|_{S_1}$ is differentiable.

In addition, π is bijective because

$$\pi(x_1, y_1, z_1) = \pi(x_2, y_2, z_2) \implies x_1 = x_2 \text{ and } y_1 = y_2$$
$$\implies z_1 = x_1^2 + y_1^2 = x_2^2 + y_2^2 = z_1 \implies (x_1, y_1, z_1) = (x_2, y_2, z_2)$$

shows injectivity and

$$\text{For arbitrary } (x, y, 0) \in S_2, \ \pi(x, y, x^2 + y^2) = (x, y, 0)$$

shows surjectivity. The inverse function of π is given by $\pi^{-1}(x, y, 0) = (x, y, x^2 + y^2)$. Since $\pi^{**} : \mathbb{R}^3 \to \mathbb{R}^3$, $\pi^{**}(x, y, z) = (x, y, x^2 + y^2)$ is differentiable and $\pi^{**}(S_2) \subset S_1$, we obtain $\pi^{-1} = \pi^{**}|_{S_2}$ is differentiable. Therefore π is a diffeomorphism.

4. Let $S_1 = \left\{(x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}$ be an ellipsoid and $S_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ be a sphere. The goal is to show that $\varphi : S_1 \to S_2$, $\varphi(x, y, z) = \left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)$ is a diffeomorphism between S_1 and S_2. Note that $\varphi^* : \mathbb{R}^3 \to \mathbb{R}^3$, $\varphi^*(x, y, z) = \left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)$ is differentiable map and $\varphi^*(S_1) \subset S_2$. Therefore $\varphi = \varphi^*|_{S_1}$ is differentiable.

It is easy to see that φ has an inverse function $\varphi^{-1} : S_2 \to S_1$, $\varphi^{-1}(x, y, z) = (ax, by, cz)$. Since $\varphi^{**} : S_2 \to S_1$, $\varphi^{**}(x, y, z) = (ax, by, cz)$ is differentiable and $\varphi^{**}(S_2) \subset S_1$, we obtain $\varphi^{-1} = \varphi^{**}|_{S_2}$ is differentiable and that φ is a diffeomorphism.

5. Let $d^* : \mathbb{R}^3 - \{\mathbf{p}_0\} \to \mathbb{R}$, $d^*(\mathbf{p}) = |\mathbf{p} - \mathbf{p}_0|$. Then d^* is a differentiable function. Hence $d = d^*|_S$ is also differentiable.

6. Suppose that a continuous map $\varphi : V_1 \to S_1$ of an open set V_1 of a regular surface S_1 to a regular surface S_2 is differentiable at $\mathbf{p} \in V_1$. That is, given parametrizations $\mathbf{x}_1 : U_1 \to S_1$, $\mathbf{x}_2 : U_2 \to S_2$ with $\mathbf{p} \in \mathbf{x}_1(U_1)$ and $\varphi(\mathbf{x}_1(U_1)) \subset \mathbf{x}_2(U_2)$, the map $\mathbf{x}_2^{-1} \circ \varphi \circ \mathbf{x}_1 : U_1 \to U_2$ is differentiable at $\mathbf{q} = \mathbf{x}_1^{-1}(\mathbf{p})$.

   Now assume that $\mathbf{y}_1 : V_1 \to S_1$, $\mathbf{y}_2 : V_2 \to S^2$ is another parametrization for S_1 and S_2 with $\mathbf{p} \in \mathbf{y}_1(V_1)$ and $\varphi(\mathbf{y}_1(V_1)) \subset \mathbf{y}_2(V_2)$. Let $W_1 = \mathbf{x}_1(U_1) \cap \mathbf{y}_1(V_1)$ and $W_2 = \mathbf{x}_2(U_2) \cap \mathbf{y}_2(V_2)$, then the change of coordinates

   $$\mathbf{h}_1 = \mathbf{x}_1^{-1} \circ \mathbf{y}_1 : \mathbf{y}_1^{-1}(W_1) \to \mathbf{x}_1^{-1}(W_1), \quad \mathbf{h}_2 = \mathbf{x}_2^{-1} \circ \mathbf{y}_2 : \mathbf{y}_2^{-1}(W_2) \to \mathbf{x}_2^{-1}(W_2)$$

   are diffeomorphisms. Therefore

   $$\mathbf{y}_2^{-1} \circ \varphi \circ \mathbf{y}_1 = (\mathbf{h}_2^{-1} \circ \mathbf{x}_2^{-1}) \circ \varphi \circ (\mathbf{x}_1 \circ \mathbf{h}_1) = \mathbf{h}_2^{-1} \circ (\mathbf{x}_2^{-1} \circ \varphi \circ \mathbf{x}_1) \circ \mathbf{h}_1$$

   is also differentiable, whence the asserted independence.

7. (i) reflexivity: We define $\mathrm{id}_{\mathbb{R}^3} : \mathbb{R}^3 \to \mathbb{R}^3$ as $\mathrm{id}_{\mathbb{R}^3}(x, y, z) = (x, y, z)$. Since $\mathrm{id}_{\mathbb{R}^3}$ is differentiable and $\mathrm{id}_{\mathbb{R}^3}(S_1) \subset S_1$, we have that $\mathrm{id}_{S_1} = \mathrm{id}_{\mathbb{R}^3}|_{S_1}$ is differentiable. Moreover, $\left(\mathrm{id}_{S_1}\right)^{-1} = \mathrm{id}_{S_1}$, so id_{S_1} is a differmorphism and S_1 is diffeomorphic to S_1.

   (ii) symmetry: Suppose S_1 is diffeomorphic to S_2, that is, there is a diffeomorphism $\varphi : S_1 \to S_2$. Now $\varphi^{-1} : S_2 \to S_1$ is also diffeomorphism because φ^{-1} and $\left(\varphi^{-1}\right)^{-1} = \varphi$ is differentiable. Hence S_2 is diffeomorphic to S_1.

   (iii) transitivity: Suppose S_1 is diffeomorphic to S_2 and S_2 is diffeomorphic to S_3. Then there exist diffeomorphisms $\varphi : S_1 \to S_2$ and $\psi : S_2 \to S_3$. Since $\varphi \circ \psi$ and $(\varphi \circ \psi)^{-1} = \psi^{-1} \circ \varphi^{-1}$ are differentiable, S_1 is diffeomorphic to S_3.

8. The half-line l is given by $l(t) = (tx, ty, z)$, $t > 0$. As $\mathbf{F}(\mathbf{p}) = l \cap H$, we obtain $t^2 x^2 + t^2 y^2 - x^2 = 1$, so $t = \sqrt{(1 + z^2)/(x^2 + y^2)}$. Therefore

   $$\mathbf{F}(x, y, z) = (tx, ty, z) = \left(x\sqrt{\frac{1 + z^2}{x^2 + y^2}}, \; y\sqrt{\frac{1 + z^2}{x^2 + y^2}}, \; z \right).$$

   Since $\mathbf{F}^* : \mathbb{R}^3 - \{0\} \to \mathbb{R}^3$ defined as $\mathbf{F}^*(x, y, z) = \left(x\sqrt{\frac{1+z^2}{x^2+y^2}}, \; y\sqrt{\frac{1+z^2}{x^2+y^2}}, \; z \right)$ is differentiable, we can conclude that $\mathbf{F} = \mathbf{F}^*|_S$ is also differentiable.

9. **a. Differentiable function on a regular curve.** Let $f : V \subset C \to \mathbb{R}$ be a function defined in an open subset V of a regular curve C. Then f is said to be differentiable at $\mathbf{p} \in V$ if, for some parametrization $\alpha : I \subset \mathbb{R} \to C$ with $\mathbf{p} \in \alpha(I) \subset V$, the composition $f \circ \alpha : I \subset \mathbb{R} \to \mathbb{R}$ is differentiable at $\alpha^{-1}(\mathbf{p})$. f is differentiable in V if it is differentiable at all points of V.

   Differentiable function between regular curves. A continuous map $\varphi : V_1 \subset C_1 \to C_2$ of an open set V_1 of a regular curve C_1 to a regular curve C_2 is said to be differentiable at $\mathbf{p} \in V_1$ if, given parametrizations

   $$\alpha_1 : I_1 \subset \mathbb{R} \to C_1, \quad \alpha_2 : I_2 \subset \mathbb{R} \to C_2$$

   with $\mathbf{p} \in \alpha_1(I_1)$ and $\varphi(\alpha_1(I_1)) \subset \alpha_2(I_2)$, the map

   $$\alpha_2^{-1} \circ \varphi \circ \alpha_1 : I_1 \to I_2$$

   is differentiable at $q = \alpha_1^{-1}(\mathbf{p})$. φ is differentiable in V_1 if it is differentiable at all points of V_1.

   Note that we need to prove that both definitions do not depend on the choice of the parametrization α for the definitions to make sense.

   b. Observe that $\mathbf{E}$ is a mapping between two regular curves $C_1 = \mathbb{R} \times \{0\}$ and $C_2 = \mathbb{S}^1$. Now choose arbitrary $\mathbf{p} \in C_1$ and given parametrizations $\alpha_1 : I_1 \subset \mathbb{R} \to C_1$ and $\alpha_2 : I_2 \subset \mathbb{R} \to C_2$, with following property $\mathbf{p} \in \alpha_1(I_1)$ and $\mathbf{E}(\alpha_1(I_1)) \subset \alpha_2(I_2)$. Then we have to show that $\alpha_2^{-1} \circ \mathbf{E} \circ \alpha_1$ is differentiable at $q = \alpha_1^{-1}(\mathbf{p})$. To achieve this, we proceed similarly to argument in proof of Prop. 1.
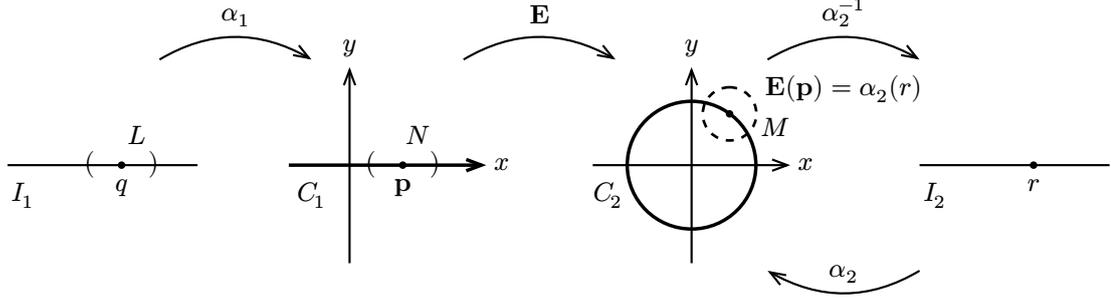
Let $r = (\alpha_2^{-1} \circ \mathbf{E} \circ \alpha_1)(q) = (\alpha_2^{-1} \circ \mathbf{E})(\mathbf{p})$. Since $\alpha_2(t) = (x(t), y(t))$ is a parametrization, we can assume, by renaming the axes if necessary, that $dx/dt(r) \neq 0$. We extend α_2 to a map $\mathbf{F} : I_2 \times \mathbb{R} \to \mathbb{R}^2$ defined by

$$\mathbf{F}(t, u) = (x(t), y(t) + u), \quad t \in I_2, u \in \mathbb{R}.$$

It is clear that $\mathbf{F}$ is differentiable and that the restriction $\mathbf{F}|_{I_2 \times \{0\}} = \alpha_2$. Calculating the determinant of the differential $d\mathbf{F_q}$,

$$\left| \begin{pmatrix} \frac{dx}{dt} & 0 \\ \frac{dy}{dt} & 1 \end{pmatrix} \right| = \frac{dx}{dt}(r) \neq 0.$$

It is possible therefore to apply the inverse function theorem, which guarantees the existence of a neighborhood M of $\alpha_2(r)$ in $\mathbb{R}^2$ such that $\mathbf{F}^{-1}$ exists and is differentiable in M.

By the continuity of $\mathbf{E}$, there exists a neighborhood N of $\mathbf{p}$ in C_1 such that $\mathbf{E}(N) \subset M$. Furthermore, by the continuity of α_1, there exists a neighborhood L of q in I_1 such that $\alpha_1(L) \subset N$. Notice that, restricted to L,

$$(\alpha_2^{-1} \circ \mathbf{E} \circ \alpha_1)|_L = \mathbf{F}^{-1} \circ \mathbf{E} \circ \alpha_1|_L$$

is a composition of differentiable maps. Thus, we can apply the chain rule for maps and conclude that $\mathbf{E}$ is differentiable at $\mathbf{p}$. Since $\mathbf{p}$ is arbitrary, $\mathbf{E}$ is differentiable on C_1.

10. The condition is that the curve C must meet the axis r orthogonally at points $\mathbf{p}$ and $\mathbf{q}$. We shall take xz plane as the plane of the curve and the z axis as the rotation axis. Let

$$x = f(v), \quad z = g(v), \quad a \leq v \leq b, \quad f(v) > 0 \text{ for } a < v < b, \quad f(v) = 0 \text{ for } v = a, b.$$

The region obtained by rotating the part of $a < v < b$ is regular surface since $f(v) > 0$. Hence it is sufficient to check regularity at the points $\mathbf{p}$ and $\mathbf{q}$. We consider the point $\mathbf{p}$. By Prop. 3 of Sec. 2-2, there exists a neighborhood V of $\mathbf{p}$ in S such that V is the graph of a differentiable function which has one of the three forms $z = h(x, y), y = i(x, z), x = j(y, z)$. Since S is a surface of revolution, it must be represented by $z = h(x, y)$. Then for every $v \in [a, b]$ and $\theta \in [0, 2\pi)$, we have

$$g(v) = h(f(v) \cos \theta, f(v) \sin \theta).$$

Since h is differentiable, we can write h as

$$h(x, y) = h(0, 0) + h_x(0, 0)x + h_y(0, 0)y + \varepsilon(x, y)\sqrt{x^2 + y^2}$$

where $\varepsilon(x, y) \to 0$ as $(x, y) \to (0, 0)$. Then put $x = f(v) \cos \theta, y = f(v) \sin \theta$ to obtain

$$g(v) = g(a) + h_x(0, 0)f(v) \cos \theta + h_y(0, 0)f(v) \sin \theta + \varepsilon(f(v) \cos \theta, f(v) \sin \theta)|f(v)|.$$

In consequence,

$$\frac{g(v) - g(a)}{v - a} = h_x(0, 0)\frac{f(v)}{v - a} \cos \theta + h_y(0, 0)\frac{f(v)}{v - a} \sin \theta + \varepsilon(f(v) \cos \theta, f(v) \sin \theta)\frac{|f(v)|}{v - a}.$$

Taking the limit $v \to a^+$, we obtain $g'(a) = \big(h_x(0, 0) \cos \theta + h_y(0, 0) \sin \theta\big)f'(a)$. If $f'(a) = 0$, then $g'(a) = 0$, so it contradicts the regularity of the generating curve. So $f'(a) = 0$. "To ensure the right-hand side yields a constant value $g'(a)$ regardless of θ, it must be $h_x(0, 0) = h_y(0, 0) = 0$, and $g'(a) = 0$. Therefore the curve C meets the r-axis orthogonally at points $\mathbf{p}$ and $\mathbf{q}$.

The conditions mentioned above are sufficient for the surface of revolution to have a well-defined tangent plane at every point. However, to satisfy the book's definition of a regular surface (having a parametrization which has continuous partial derivatives of all orders), additional conditions are required; specifically, it seems like

- $g'(a) = g'(b) = 0$,
- $f'(a) \neq 0, f'(b) \neq 0$,
- $g^{(2k-1)}(a) = g^{(2k-1)}(b) = 0$ for every $k \in \mathbb{N}$,
- $f^{(2k)}(a) = f^{(2k)}(b) = 0$ for every $k \in \mathbb{N}$.

Nevertheless, I have not yet been able to provide a rigorous proof for this.

11. Let $R_{\theta,r}$ denote the rotation by an angle θ about the axis r. Since

$$R_{\theta,r}(\mathbf{p} + \mathbf{q}) = R_{\theta,r}(\mathbf{p}) + R_{\theta,r}(\mathbf{q}), \quad R_{\theta,r}(c\mathbf{p}) = cR_{\theta,r}(\mathbf{p}), \quad \mathbf{p}, \mathbf{q} \in \mathbb{R}^3$$

$R_{\theta,r}$ is a linear map and is therefore differentiable. Moreover, as S is a surface of revolution about axis r, it follows that $R_{\theta,r}(S) \subset S$. Thus the restriction $\varphi = R_{\theta,r}|_S$ and its inverse $\varphi^{-1} = (R_{\theta,r})^{-1}|_S = R_{-\theta,r}|_S$ are differentiable. Consequently, φ is a diffeomorphism of S.

12. a. Since the straight line passing through a moving point $\alpha(t) \in C$ and the fixed point $\mathbf{0}$ is given by $\{v\alpha(t) \mid v \in R\}$, we can represent Σ by the trace of a parametrized surface

$$\mathbf{x} : U \to \Sigma, \quad \mathbf{x}(v, t) = v\alpha(t),$$

where $U = \{(v, t) \mid v \in \mathbb{R}, \ t \in (a, b)\}$.

b. For a given point $\mathbf{q} \in U$ to be regular, the differential $d\mathbf{x_q} : \mathbb{R}^2 \to \mathbb{R}^3$ must be one-to-one. We denote $\alpha(t) = (x(t), y(t), z(t))$, then

$$d\mathbf{x_q} = \begin{pmatrix} x(t) & vx'(t) \\ y(t) & vy'(t) \\ z(t) & vz'(t) \end{pmatrix}.$$

Observe that

$$\left(\frac{\partial(x, y)}{\partial(v, t)}, -\frac{\partial(x, z)}{\partial(v, t)}, \frac{\partial(y, z)}{\partial(v, t)} \right) = v(xy' - x'y, zx' - xz', yz' - zy') = v\alpha(t) \wedge \alpha'(t),$$

so $d\mathbf{x_q}$ is one-to-one if and only if $v\alpha(t) \wedge \alpha'(t) \neq 0$. Hence $\mathbf{x}$ is not regular at the origin ($v = 0$) and at points where the tangent passes through the origin ($\alpha(t) \parallel \alpha'(t)$).

c. For the remaining set of Σ to be a regular surface, we must first remove the points where the parametrization $\mathbf{x}$ is not regular.

Additionally, as shown in the given figure, we must remove the lines of self-intersection in Σ, because Σ might fail to satisfy the condition of being locally homeomorphic to a plane at these points. Let's examine this condition in more detail.

First, for a self-intersection to occur, there must exist distinct $t_1, t_2 \in (a, b)$ such that $\alpha(t_1) \parallel \alpha(t_2)$. Since the normal vector of the tangent plane at a given point $\mathbf{q} = (v, t)$ is determined by $\mathbf{x}_v \wedge \mathbf{x}_t = v\alpha(t) \wedge \alpha'(t)$, the condition for the tangent planes to be distinct (non-parallel) at the intersection is

$$v_1\alpha(t_1) \wedge \alpha'(t_1) \nparallel v_2\alpha(t_2) \wedge \alpha'(t_2).$$

Since non-zero scalars $v_1, v_2 \neq 0$ do not affect the direction, and because $\alpha(t_1) \parallel \alpha(t_2)$, this condition can be rewritten as

$$\alpha(t_1) \wedge \alpha'(t_1) \nparallel \alpha(t_1) \wedge \alpha'(t_2).$$

This is equivalent to their cross product being non-zero. Using the vector quadruple product identity, we can derive

$$\mathbf{0} \neq (\alpha(t_1) \wedge \alpha'(t_1)) \wedge (\alpha(t_1) \wedge \alpha'(t_2))$$
$$= \det(\alpha(t_1), \alpha'(t_1), \alpha'(t_2)) \cdot \alpha(t_2) - \det(\alpha(t_1), \alpha'(t_1), \alpha(t_2)) \cdot \alpha'(t_2)$$
$$= \det(\alpha(t_1), \alpha'(t_1), \alpha'(t_2)) \cdot \alpha(t_2).$$

Since $\alpha(t_2) \neq \mathbf{0}$, this condition is equivalent to

$$\det(\alpha(t_1), \alpha'(t_1), \alpha'(t_2)) \neq 0.$$

In summary, the points and lines that must be removed are:

1. The origin $\mathbf{0}$,
2. The line of non-regular points: $\{\mathbf{x}(v,t) \in \Sigma \mid v \in \mathbb{R}, t \in (a,b) \text{ such that } \alpha(t) \parallel \alpha'(t)\}$
3. The line of self-intersections:

$$\left\{ \mathbf{x}(v, t_1) \in \Sigma \ \middle| \ \begin{matrix} v \in \mathbb{R}, \ \exists\, t_1 \neq t_2 \in (a,b) \text{ such that } \alpha(t_1) \parallel \alpha(t_2) \\ \text{and } \det(\alpha(t_1), \alpha'(t_1), \alpha'(t_2)) \neq 0 \end{matrix} \right\}$$

13. We have to show that $f : V \subset S \to \mathbb{R}$ is differentiable at $\mathbf{p} \in V$ if and only if $f = f^*|_V$ for some differentiable function $f^* : V^* \to \mathbb{R}$, where $\mathbf{p}$ is contained in an open set $V^* \subset \mathbb{R}^3$.

First suppose that f is such a restriction. Since f^* is differentiable, $f = f^*|_V$ is also differentiable.

Conversely, suppose f is differentiable at $\mathbf{p} \in V$, that is, if $\mathbf{x} : U \to \mathbb{R}^3$ is a parametrization for S at $\mathbf{p}$, then $f \circ \mathbf{x}$ is differentiable at $\mathbf{x}^{-1}(\mathbf{p})$. Since

$$\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v))$$

is a parametrization, we can assume, by renaming the axes if necessary, that $\frac{\partial(x,y)}{\partial(u,v)}(q) \neq 0$. We extend $\mathbf{x}$ to a map $F : U \times \mathbb{R} \to \mathbb{R}^3$ as

$$\mathbf{F}(u,v,t) = \mathbf{x}(u,v) + t\hat{\mathbf{z}} = (x(u,v), y(u,v), z(u,v) + t).$$

We can see that $\mathbf{F}$ is differentiable and $\mathbf{F}|_{U \times \{0\}} = \mathbf{x}$. Since

$$\det(d\mathbf{F_q}) = \det \begin{pmatrix} \frac{dx}{du} & \frac{dx}{dv} & 0 \\ \frac{dy}{du} & \frac{dy}{dv} & 0 \\ \frac{dz}{du} & \frac{dz}{dv} & 1 \end{pmatrix} = \frac{\partial(x,y)}{\partial(u,v)}(q) \neq 0,$$

we can apply the inverse theorem, which guarantees the existence of a neighborhood W of $\mathbf{p}$ in $\mathbb{R}^3$ such that $\mathbf{F}^{-1}$ is a diffeomorphism. Define $g : W \to \mathbb{R}$ by

$$g = f \circ \mathbf{x} \circ \pi \circ \mathbf{F}^{-1},$$

where $\pi : U \times \mathbb{R} \to U$, $\pi(u,v,t) = (u,v)$ is the natural projection. Then g is differentiable at $\mathbf{p}$, because it is a composition of differentiable functions. Now observe that f is the restriction to $W \cap S$ of a differentiable function g, i.e., $f = g|_{W \cap S}$.

14. First suppose that A is a regular surface. Given point $\mathbf{p} \in A$, we wish to find a neighborhood of $\mathbf{p}$ in S that is contained in A. Since S is a regular surface, there exists a neighborhood $V \subset \mathbb{R}^3$ of $\mathbf{p}$ and parametrization

$$\mathbf{x} : U \to V \cap S,$$

where $U \subset \mathbb{R}^2$ is open. Since A is a regular surface, there exists a neighborhood $V' \subset \mathbb{R}^3$ of $\mathbf{p}$ and parametrization

$$\mathbf{y} : U' \to V' \cap A,$$

where $U' \subset \mathbb{R}^2$ is open. By shrinking the neighborhoods if necessary, we can assume that $V' \cap A \subset V \cap S$. Now consider the map $\mathbf{h} : U' \to U$ defined by

$$\mathbf{h} = \mathbf{x}^{-1} \circ \mathbf{y}.$$

Since $\mathbf{y}(U') = V' \cap A \subset V \cap S$ and $\mathbf{x}$ is a homeomorphism onto $V \cap S$, this map is well-defined. Applying the chain rule to $\mathbf{y} = \mathbf{x} \circ \mathbf{h}$, we obtain

$$d\mathbf{y_q} = d\mathbf{x_{h(q)}} \circ d\mathbf{h_q}$$

for $\mathbf{q} = \mathbf{y}^{-1}(\mathbf{p}) \in U'$. Since $\mathbf{y}$ is a parametrization, $d\mathbf{y_q}$ is one-to-one. Thus $d\mathbf{h_q}$ is also one-to-one, that is, $d\mathbf{h_q} : \mathbb{R}^2 \to \mathbb{R}^2$ is an isomorphism. By the inverse function theorem, there exists a neighborhood N of $\mathbf{q}$ in U' and N of $\mathbf{h}(\mathbf{q})$ in U such that $\mathbf{h} : M \to N$ has a differentiable inverse $\mathbf{h}^{-1} : N \to M$. Specifically, $\mathbf{h}$ maps an open neighborhood of $\mathbf{q}$ in U' to an open neighborhood of $\mathbf{h}(\mathbf{q})$ in U.

Now let $M' \subset M$ be an open neighborhood of $\mathbf{q}$. Then $\mathbf{h}(M')$ is open in U. Moreover, because $\mathbf{x}$ is a homeomorphism, it maps an open set $\mathbf{h}(M')$ to an open set in $V \cap S$, that is, an open set in S. Therefore

$\mathbf{x}(\mathbf{h}(M')) = \mathbf{y}(M')$ is a neighborhood of $\mathbf{p}$ in S. Since $\mathbf{y}(M') \subset A$, we found a neighborhood of $\mathbf{p}$ in S that is entirely contained in A. Therefore A is open in S.

Conversely, suppose that A is open in S. Given point $\mathbf{p} \in A \subset S$, since S is a regular surface, there exists a neighborhood V in $\mathbb{R}^3$ and a parametrization $\mathbf{x} : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$. Observe that $V \cap A$ is open in S. Now we show that

$$\mathbf{y} = \mathbf{x}|_{\mathbf{x}^{-1}(V \cap A)} : \mathbf{x}^{-1}(V \cap A) \to V \cap A.$$

is a parametrization for A at $\mathbf{p}$. (i) $\mathbf{y}$ is differentiable since $\mathbf{x}$ is differentiable. (ii) $\mathbf{y}$ is homeomorphism, being restriction of the homeomorphism $\mathbf{x}$ to the open set $\mathbf{x}^{-1}(V \cap A)$. (iii) the differential $d\mathbf{y_q}$ is one-to-one for each $\mathbf{q} \in \mathbf{x}^{-1}(V \cap A)$, since $d\mathbf{x_q}$ is injective for each $\mathbf{q} \in U$. Therefore A is a regular surface.

15. a. Let $r \in \beta^{-1}(W)$ and $q = h(r)$. Since $\alpha(t) = (x(t), y(t), z(t))$ is a parametrization, we can assume, by renaming the axes if necessary, that $dx/dt(q) \neq 0$. We extend α to a map $\mathbf{F} : I \times \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$\mathbf{F}(t, u, v) = (x(t), y(t) + u, z(t) + v), \quad t \in I, \ u, v \in \mathbb{R}.$$

It is clear that $\mathbf{F}$ is differentiable and $\mathbf{F}|_{I \times \{0\} \times \{0\}} = \alpha$. Calculating the determinant of the differential $d\mathbf{F_q}$, we obtain

$$\det(d\mathbf{F_q}) = \det \begin{pmatrix} \frac{dx}{dt} & 0 & 0 \\ \frac{dy}{dt} & 1 & 0 \\ \frac{dz}{dt} & 0 & 1 \end{pmatrix} = \frac{dx}{dt}(q) \neq 0.$$

It is possible therefore to apply inverse function theorem, which guarantees the existence of a neighborhood M of $\alpha(q)$ in $\mathbb{R}^3$ such that $\mathbf{F}^{-1}$ exists and is differentiable in M.

By the continuity of β, there exists neighborhood N of r in J such that $\beta(N) \subset M$. Notice that, restricted to N, $h|_N = \mathbf{F}^{-1} \circ \beta|_N$ is a composition of differentiable maps. Thus, we can apply the chain rules for maps and conclude that h is differentiable at r. Since r is arbitrary, h is differentiable on $\beta^{-1}(W)$.

Exactly the same argument can be applied to show that the map h^{-1} is differentiable, and so h is a diffeomorphism.

b. Let $t = h(\tau)$. Differentiating both sides of $\alpha(t) = \beta(\tau)$ with respect to t, then we obtain

$$\alpha'(t) = \beta'(\tau)\frac{d\tau}{dt}.$$

Note that $d\tau/dt \neq 0$ doesn't change the sign. If not, there exists t such that $d\tau/dt = 0$ by the intermediate value theorem. Then we have $\alpha'(t) = 0$, which contradicts the regularity of α. Hence there exists a sign $\mu = \pm 1$ such that

$$\frac{d\tau}{dt} = \mu \frac{|\alpha'(t)|}{|\beta'(\tau)|}$$

for every $t \in I$. Therefore

$$\left| \int_{\tau_0}^{\tau} |\beta'(\tau)| \, d\tau \right| = \left| \int_{t_0}^{t} |\beta'(\tau)| \frac{d\tau}{dt} \, dt \right| = \left| \int_{t_0}^{t} \mu |\alpha'(t)| \, dt \right| = |\mu| \left| \int_{t_0}^{t} |\alpha'(t)| \, dt \right| = \left| \int_{t_0}^{t} |\alpha'(t)| \, dt \right|.$$

16. We first assert that $\mathbf{F}$ is differentiable on $\mathbb{S}^2 - \{N\}$. To show this, for each $\mathbf{p} \in \mathbb{S}^2 - \{N\}$, and given parametrization $\mathbf{x}_1$ and $\mathbf{x}_2$ at $\mathbf{p}$, we must show that

$$\mathbf{x}_2^{-1} \circ \mathbf{F} \circ \mathbf{x}_1 = \mathbf{x}_2^{-1} \circ \pi_N^{-1} \circ P \circ \pi_N \circ \mathbf{x}_1$$

is differentiable at $\mathbf{x}_1^{-1}(\mathbf{p})$. Since π_N^{-1} is a parametrization for $\mathbb{S}^2$, the change of coordinates

$$\pi_N \circ \mathbf{x}_1 = \left(\pi_N^{-1}\right)^{-1} \circ \mathbf{x}_1 \quad \text{and} \quad \pi_N \circ \mathbf{x}_2 = \left(\pi_N^{-1}\right)^{-1} \circ \mathbf{x}_2$$

are diffeomorphisms. Therefore

$$\mathbf{x}_2^{-1} \circ \pi_N^{-1} \circ P \circ \pi_N \circ \mathbf{x}_1 = \left(\pi_N \circ \mathbf{x}_1\right)^{-1} \circ P \circ \left(\pi_N \circ \mathbf{x}_1\right)$$

is differentiable at $\mathbf{x}_1^{-1}(\mathbf{p})$. Now we show that $\mathbf{F}$ is differentiable at N. Consider the stereographic projection π_S from the south pole $S = (0, 0, -1)$ to $\mathbb{C}$. (Here, we identify the plane $z = 1$ with $\mathbb{C}$.) Define the complex polynomial $Q : \mathbb{C} \to \mathbb{C}$ as

$$Q(\zeta) = 4/\overline{P\left(4/\overline{\zeta}\right)} = \frac{4\zeta^n}{4^n\overline{a_0} + 4^{n-1}\overline{a_1}\zeta + \cdots + \overline{a_n}\zeta^n},$$

and we shall show that the equation

$$\mathbf{F} = \pi_S^{-1} \circ Q \circ \pi_S$$

holds in the neighborhood of N. For $\mathbf{p} = N$, we can easily check that the equation holds, because $\mathbf{F}(N) = N$ and $\pi_S^{-1}(Q(\pi_S(N))) = \pi_S^{-1}(Q(0)) = \pi_S^{-1}(0) = N$. Now suppose that $\mathbf{p} \neq N$. We have to show that

$$\pi_N^{-1} \circ P \circ \pi_N = \pi_S^{-1} \circ Q \circ \pi_S$$

in the neighborhood of N. Observe that

$$\pi_N^{-1}(u, v) = \left(\frac{4u}{u^2 + v^2 + 4}, \frac{4v}{u^2 + v^2 + 4}, \frac{u^2 + v^2 - 4}{u^2 + v^2 + 4} \right),$$

$$\pi_S^{-1}(u, v) = \left(\frac{4u}{u^2 + v^2 + 4}, \frac{4v}{u^2 + v^2 + 4}, \frac{4 - u^2 - v^2}{u^2 + v^2 + 4} \right).$$

Let $\xi = \left(\pi_N \circ \pi_S^{-1}\right)(\zeta)$, then $\pi_N^{-1}(\xi) = \pi_S^{-1}(\zeta)$. By a slight abuse of notation, we have

$$\left(\frac{4\xi}{|\xi|^2 + 4}, \frac{|\xi|^2 - 4}{|\xi|^2 + 4} \right) = \pi_N^{-1}(\xi) = \pi_S^{-1}(\zeta) = \left(\frac{4\zeta}{|\zeta|^2 + 4}, \frac{4 - |\zeta|^2}{|\zeta|^2 + 4} \right).$$

From $(|\xi|^2 - 4)/(|\xi|^2 + 4) = (4 - |\zeta|^2)/(|\zeta|^2 + 4)$, we obtain $|\xi| = 4/|\zeta|$, then from $(4\xi)/(|\xi|^2 + 4) = (4\zeta)/(|\zeta|^2 + 4)$, we get $\xi = 4/\overline{\zeta}$. By the definition of Q,

$$\left(\pi_S \circ \pi_N^{-1} \circ P \circ \pi_N \circ \pi_S^{-1}\right)(\zeta) = \left(\left(\pi_N \circ \pi_S^{-1}\right)^{-1} \circ P \circ \left(\pi_N \circ \pi_S^{-1}\right)\right)(\zeta) = 4/\overline{P\left(4/\overline{\zeta}\right)} = Q(\zeta),$$

which means $\pi_N^{-1} \circ P \circ \pi_N = \pi_S^{-1} \circ Q \circ \pi_S$. Therefore we have $\mathbf{F} = \pi_S^{-1} \circ Q \circ \pi_S$ in the neighborhood of N. Since Q is differentiable at $\zeta = 0$, $\mathbf{F} = \pi_S^{-1} \circ Q \circ \pi_S$ is differentiable at N.

2-4. The Tangent Plane; The Differential of a Map

1. The regular surface S is given by $S = \{(x, y, z) \mid f(x, y, z) = 0\}$. Let $\mathbf{p} = (x, y, z)$ and $\mathbf{p}_0 = (x_0, y_0, z_0)$. Suppose $\mathbf{w} = \mathbf{p} - \mathbf{p}_0$ is a tangent vector to S at a point $\mathbf{p}_0$, that is, $\mathbf{w} = \alpha'(0)$ where $\alpha : (-\varepsilon, \varepsilon) \to S$ is differentiable and $\alpha(0) = \mathbf{p}_0$. Write the curve as $\alpha(t) = (x_\alpha(t), y_\alpha(t), z_\alpha(t))$, then we have

$$(f \circ \alpha)(t) = f(\alpha(t)) = f(x_\alpha(t), y_\alpha(t), z_\alpha(t)) = 0$$

because $\alpha(t) \in S$ for every $t \in (-\varepsilon, \varepsilon)$. Now differentiate both sides with respect to t, then

$$f_x(\alpha(t))x_{\alpha'}(t) + f_y(\alpha(t))y_{\alpha'}(t) + f_z(\alpha(t))z_{\alpha'}(t) = 0.$$

Evaluating this at $t = 0$, we obtain

$$f_x(\mathbf{p}_0)x_{\alpha'}(0) + f_y(\mathbf{p}_0)y_{\alpha'}(0) + f_z(\mathbf{p}_0)z_{\alpha'}(0) = 0.$$

Since $(x_{\alpha'}(0), y_{\alpha'}(0), z_{\alpha'}(0)) = \alpha'(0) = \mathbf{w} = \mathbf{p} - \mathbf{p}_0 = (x - x_0, y - y_0, z - z_0)$, we have

$$f_x(\mathbf{p}_0)(x - x_0) + f_y(\mathbf{p}_0)(y - y_0) + f_z(\mathbf{p}_0)(z - z_0) = 0.$$

Conversely, suppose this equation holds, and we will show that $(x - x_0, y - y_0, z - z_0)$ is a tangent vector to S at $\mathbf{p}_0$. Since 0 is a regular value of f, it is possible to assume, by renaming the axes if necessary, that $f_z \neq 0$ at $\mathbf{p}_0$. Based on the argument from Prop. 2 of Sec. 2-2, the curve can be expressed by the equation $z = g(x, y)$ in the neighborhood of $\mathbf{p}_0$. Now we define the curve $\alpha : (-\varepsilon, \varepsilon) \to S$ as

$$\alpha(t) = (x_0 + (x - x_0)t, \ y_0 + (y - y_0)t, \ g(x_0 + (x - x_0)t, y_0 + (y - y_0)t)).$$

Since $f(x, y, g(x, y)) = 0$ in the neighborhood of $\mathbf{p_0}$, differentiating both sides with respect to x and y respectively, evaluating these at (x_0, y_0) yields

$$f_x(\mathbf{p_0}) + f_z(\mathbf{p_0})g_x(x_0, y_0) = 0, \quad f_y(\mathbf{p_0}) + f_z(\mathbf{p_0})g_y(x_0, y_0) = 0.$$

Recall that $f_x(\mathbf{p_0})(x - x_0) + f_y(\mathbf{p_0})(y - y_0) + f_z(\mathbf{p_0})(z - z_0) = 0$, we have

$$\alpha'(0) = \left(x - x_0, y - y_0, g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0)\right)$$

$$= \left(x - x_0, y - y_0, \left(-\frac{f_x(\mathbf{p_0})}{f_z(\mathbf{p_0})}\right)(x - x_0) + \left(-\frac{f_y(\mathbf{p_0})}{f_z(\mathbf{p_0})}\right)(y - y_0)\right)$$

$$= (x - x_0, y - y_0, z - z_0).$$

Therefore $(x - x_0, y - y_0, z - z_0)$ is a tangent vector to S at $\mathbf{p_0}$.

2. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ as $f(x, y, z) = x^2 + y^2 - z^2 - 1$. Since $df_\mathbf{q} = (2x, 2y, -2x) = (0, 0, 0)$ implies $(x, y, z) = (0, 0, 0)$, the critical value of f is $f(0, 0, 0) = -1$. Thus the given surface $f^{-1}(0)$ is a regular surface by Prop. 2 of Sec. 2-2. By Exercise 1, the equation of the tangent plane at (x_0, y_0, z_0) is given by $2x_0(x - x_0) + 2y_0(y - y_0) - 2z_0(z - z_0) = 0$. Put $z_0 = 0$ then we obtain

$$x_0(x - x_0) + y_0(y - y_0) = 0.$$

The normal vector of the tangent plane is $(x_0, y_0, 0)$, which is perpendicular to $\hat{\mathbf{z}} = (0, 0, 1)$. Therefore the tangent planes are all parallel to the z axis.

3. Let $g(x, y, z) = f(x, y) - z$. Since $dg_\mathbf{q} = (f_x, f_y, -1) \neq (0, 0, 0)$, we see that $S = g^{-1}(0)$ is a regular surface. By Exercise 1, the equation of the tangent plane at (x_0, y_0, z_0) is given by

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0,$$

where $z_0 = f(x_0, y_0)$, which is equivalent to

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

To see that the graph of the differential $df_\mathbf{p}$ is the tangent plane, choose arbitrary $\mathbf{w} \in T_{\mathbf{p_0}}(S)$ and find the differentiable parametrized curve $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = \mathbf{p_0}$ and $\alpha'(0) = \mathbf{w}$. We denote $\alpha(t) = (x_\alpha(t), y_\alpha(t))$. Then for $\beta = f \circ \alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}$, we have

$$df_{\mathbf{p_0}}(\mathbf{w}) = \beta'(0) = \frac{d}{dt} f(x_\alpha(t), y_\alpha(t))\Big|_{t=0} = f_x(\mathbf{p_0})x'_\alpha(0) + f_y(\mathbf{p_0})y'_\alpha(0).$$

Denote $\mathbf{w} = (x'_\alpha(0), y'_\alpha(0)) = (x_\mathbf{w}, y_\mathbf{w})$, then the graph of the differential $df_{\mathbf{p_0}}$ is

$$\left\{\left(x_\mathbf{w}, y_\mathbf{w}, df_{\mathbf{p_0}}(x_\mathbf{w}, y_\mathbf{w})\right) \mid f_x(\mathbf{p_0})x_\mathbf{w} + f_y(\mathbf{p_0})y_\mathbf{w} = df_{\mathbf{p_0}}(x_\mathbf{w}, y_\mathbf{w})\right\}$$

$$= \left\{(x - x_0, y - y_0, z - z_0) \mid f_x(\mathbf{p_0})(x - x_0) + f_y(\mathbf{p_0})(y - y_0) = z - z_0\right\}$$

$$= \left\{(x, y, z) \mid f_x(\mathbf{p_0})(x - x_0) + f_y(\mathbf{p_0})(y - y_0) = z - z_0\right\} - (x_0, y_0, z_0),$$

which is the translation of the tangent plane at $\mathbf{p_0}$.

4. Let $g(x, y) = x f(y/x)$. Since $g : \mathbb{R}^2 - \{0\} \to \mathbb{R}$ is the differentiable function defined on an open set, $S = \{(x, y, g(x, y)) \mid (x, y) \in \mathbb{R}^2 - \{0\}\}$ is a regular surface by Prop. 1 of Sec. 2-2. By Exercise 3, the tangent plane at $\mathbf{p_0} = (x_0, y_0, z_0)$ is given by

$$z = g(x_0, y_0) + g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0)$$

$$= x_0 f\left(\frac{y_0}{x_0}\right) + \left(f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0} f'\left(\frac{y_0}{x_0}\right)\right)(x - x_0) + f'\left(\frac{y_0}{x_0}\right)(y - y_0).$$

Now put $x = y = 0$ then we obtain

$$z = x_0 f\left(\frac{y_0}{x_0}\right) + \left(f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0} f'\left(\frac{y_0}{x_0}\right)\right)(-x_0) + f'\left(\frac{y_0}{x_0}\right)(-y_0) = 0.$$

Hence the tangent planes all pass through the origin $(0, 0, 0)$.

5. We first show that the normal vector of the tangent plane of S at $\mathbf{p} = \mathbf{x}(u,v)$ is given by $\mathbf{x}_u(u,v) \wedge \mathbf{x}_v(u,v)$. if we define $\alpha : (-\varepsilon, \varepsilon) \to S$ as $\alpha(t) = \mathbf{x}(u+t, v)$, then we have $\alpha'(0) = \mathbf{x}_u(u,v)$ so $\mathbf{x}_u(u,v) \in T_{\mathbf{p}}(S)$. Similarly, $\mathbf{x}_v(u,v) \in T_{\mathbf{p}}(S)$. Since $\mathbf{x}$ is a parametrization for S,

$$d\mathbf{x}_{(u,v)} = \left(\mathbf{x}_u \middle| \mathbf{x}_v \right)$$

is one-to-one, that is, $\mathbf{x}_u \wedge \mathbf{x}_v \neq \mathbf{0}$ at (u,v). Therefore $\mathbf{x}_u \wedge \mathbf{x}_v$ is the normal vector of $T_{\mathbf{p}}(S)$.

Now consider the fixed coordinate curve $u = u_0$. The parametrization of the surface is given by $\mathbf{x}(u,v) = \alpha_1(u) + \alpha_2(v)$, thus the normal vector of the tangent plane at $\mathbf{x}(u_0, v)$ is $\alpha_1'(u_0) \wedge \alpha_2'(v)$, which is perpendicular to $\alpha_1'(u_0)$ for every v. Therefore the tangent planes along a fixed coordinate curve $u = u_0$ are all parallel to a line.

6. The tangent plane of the surface S at $\mathbf{p} = \mathbf{x}(t,v)$ contains two vectors $\mathbf{x}_t = \alpha'(t) + v\alpha''(t)$ and $\mathbf{x}_v = \alpha'(t)$. Observe that $\mathbf{x}_t \wedge \mathbf{x}_v = v\alpha''(t) \wedge \alpha'(t)$. Since the curvature of the parametrized curve α is everywhere nonzero, we have $k(t) = |\alpha'(t) \wedge \alpha''(t)|/|\alpha'(t)|^3 \neq 0$, therefore the normal vector of $T_{\mathbf{p}}(S)$ is $\mathbf{x}_t \wedge \mathbf{x}_v = v\alpha''(t) \wedge \alpha'(t) \neq \mathbf{0}$.

Now fix $t = t_0$. The equation of tangent plane at $\mathbf{p}_v = \mathbf{x}(t_0, v)$ is given by

$$\mathbf{y} \in T_{\mathbf{p}_v}(S) \iff (\alpha''(t_0) \wedge \alpha'(t_0)) \cdot (\mathbf{y} - \mathbf{x}(t_0, v)) = 0.$$

For arbitrary v_1, v_2, observe that

$$\mathbf{x}(t_0, v_2) - \mathbf{x}(t_0, v_1) = (v_2 - v_1)\alpha'(t_0) \perp \alpha''(t_0) \wedge \alpha'(t_0).$$

Therefore

$$\begin{aligned} \mathbf{y} \in T_{\mathbf{p}_{v_2}}(S) &\iff (\alpha''(t_0) \wedge \alpha'(t_0)) \cdot (\mathbf{y} - \mathbf{x}(t_0, v_2)) = 0 \\ &\iff (\alpha''(t_0) \wedge \alpha'(t_0)) \cdot (\mathbf{y} - \mathbf{x}(t_0, v_2) + \mathbf{x}(t_0, v_2) - \mathbf{x}(t_0, v_1)) = 0 \\ &\iff (\alpha''(t_0) \wedge \alpha'(t_0)) \cdot (\mathbf{y} - \mathbf{x}(t_0, v_1)) = 0 \\ &\iff \mathbf{y} \in T_{\mathbf{p}_{v_1}}(S). \end{aligned}$$

Since v_1 and v_2 are arbitrary, the tangent planes along the curve $\mathbf{x}(t_0, v)$ all coincide.

7. Given $\mathbf{w} \in T_{\mathbf{p}}(S)$, we can choose differentiable curve $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = \mathbf{p}$, $\alpha'(0) = \mathbf{w}$. Since $f(\alpha(t)) = |\alpha(t) - \mathbf{p}_0|^2 = (\alpha(t) - \mathbf{p}_0) \cdot (\alpha(t) - \mathbf{p}_0)$,

$$df_{\mathbf{p}}(\mathbf{w}) = \frac{d}{dt} f(\alpha(t)) \bigg|_{t=0} = (2\alpha'(t) \cdot (\alpha(t) - \mathbf{p}_0))|_{t=0} = 2\alpha'(0) \cdot (\alpha(0) - \mathbf{p}_0) = 2\mathbf{w} \cdot (\mathbf{p} - \mathbf{p}_0).$$

8. Since the linear map $\mathbf{L}$ is differentiable and $\mathbf{L}(S) \subset S$, $\mathbf{L}|_S$ is also differentiable. Given $\mathbf{w} \in T_{\mathbf{p}}(S)$, we can choose differentiable curve $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = \mathbf{p}$, $\alpha'(0) = \mathbf{w}$. Then

$$d\mathbf{L}_{\mathbf{p}}(\mathbf{w}) = \frac{d}{dt} \mathbf{L}(\alpha(t)) \bigg|_{t=0} = d\mathbf{L}_{\alpha(0)}(\alpha'(0)) = d\mathbf{L}_{\mathbf{p}}(\mathbf{w}) = \mathbf{L}(\mathbf{w}).$$

9. We first show that given parametrized surface is regular. (i) It is easy to see that $\mathbf{x}$ is differentiable. (iii) Since

$$d\mathbf{x}_{\mathbf{q}} = \begin{pmatrix} -v\sin u & \cos u \\ v\cos u & \sin u \\ a & 0 \end{pmatrix}, \quad \left(\frac{\partial(x,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 = (-a\cos u)^2 + (-a\sin u)^2 = a^2 \neq 0,$$

the differential $d\mathbf{x}_{\mathbf{q}}$ is one-to-one for each $\mathbf{q} \in U$. (ii) Let $(v\cos u, v\sin u, au) = (x, y, z)$. Since $u = z/a$ and the equation $(v\cos u, v\sin u) = (x, y)$ determines v, $\mathbf{x}$ is one-to-one. By Prop. 4 of Sec 2-2, $\mathbf{x}^{-1}$ is continuous. Therefore the surface S given by parametrization $\mathbf{x}$ is regular.

Now $\mathbf{x}_u = (-v\sin u, v\cos u, a)$ and $\mathbf{x}_v = (\cos u, \sin u, 0)$, so $\mathbf{x}_u \wedge \mathbf{x}_v = (-a\sin u, a\cos u, -v)$ and $\mathbf{N}(u,v) = \frac{1}{\sqrt{a^2+v^2}}(-a\sin u, a\cos u, -v)$. Let $\theta(v) \in (0, \pi/2)$ denote the angle between the tangent plane of S at $\mathbf{x}(u_0, v)$ and the z axis. If $\phi(v)$ is the angle between the unit normal vector of $T_{\mathbf{x}(u_0, v)}(S)$ and z axis, then $\sin\theta(v) = |\cos\phi(v)|$. Since

$$\cos \phi(v) = \mathbf{N}(u_0, v) \cdot (0, 0, 1) = -\frac{v}{\sqrt{a^2 + v^2}},$$

we have $\sin \theta(v) = v/\sqrt{a^2 + v^2}$ and $\cos \theta(v) = |a|/\sqrt{a^2 + v^2}$. Finally, we can see that $\tan \theta(v) = |a|/v$ is proportional to the inverse of the distance v of the point $\mathbf{x}(u_0, v)$ to the z axis.

10. Observe that

$$\begin{aligned}
\mathbf{x}_s &= \alpha'(s) + r(\mathbf{n}'(s) \cos v + \mathbf{b}'(s) \sin v) \\
&= \mathbf{t}(s) + r((-k(s)\mathbf{t}(s) - \tau(s)\mathbf{b}(s)) \cos v + \tau(s)\mathbf{n}(s) \sin v), \\
\mathbf{x}_v &= r(-\mathbf{n}(s) \sin v + \mathbf{b}(s) \cos v)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{x}_s \wedge \mathbf{x}_v &= -r\mathbf{b}(s) \sin v - r\mathbf{n}(s) \cos v \\
&\quad + r^2 \left(k(s)\mathbf{b}(s) \sin v \cos v + k(s)\mathbf{n}(s) \cos^2 v - \tau(s)\mathbf{t}(s) \sin v \cos v + \tau(s)\mathbf{t}(s) \sin v \cos v\right) \\
&= -r\mathbf{b}(s) \sin v - r\mathbf{n}(s) \cos v + r^2 k(s) \cos v(\mathbf{b}(s) \sin v + \mathbf{n}(s) \cos v) \\
&= -r(1 - rk(s) \cos v)(\mathbf{b}(s) \sin v + \mathbf{n}(s) \cos v).
\end{aligned}$$

Since $\mathbf{x}$ is a parametrization for the surface, $\mathbf{x}_s$ and $\mathbf{x}_v$ are linearly independent, that is, $\mathbf{x}_s \wedge \mathbf{x}_v \neq \mathbf{0}$. Therefore we obtain $1 - rk(s) \cos v \neq 0$. Finally, the unit normal vector is

$$\mathbf{N}(s, v) = -\operatorname{sgn}(r(1 - rk(s) \cos v))(\mathbf{n}(s) \cos v + \mathbf{b}(s) \sin v).$$

 Note If we suppose the natural situation, that is, $r > 0$ and $1/k(s) > r$, then we can conclude that $\mathbf{N}(s, v) = -(\mathbf{n}(s) \cos v + \mathbf{b}(s) \sin v)$. However, if $1/k(s) < r$, in the case where the tubular surface has a self-intersection, then $\mathbf{N}(s, v)$ might change direction.
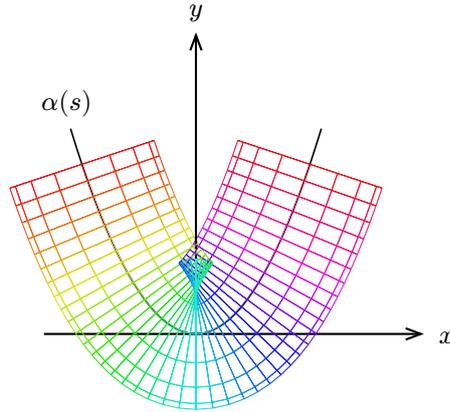
11. Observe that

$$\begin{aligned}
\mathbf{x}_u &= (f'(u) \cos v, f'(u) \sin v, g'(u)), \\
\mathbf{x}_v &= (-f(u) \sin v, f(u) \cos v, 0),
\end{aligned}$$

thus

$$\mathbf{x}_u \wedge \mathbf{x}_v = (-g'(u)f(u) \cos v, -g'(u)f(u) \sin v, f'(u)f(u)).$$

Note that $\mathbf{x}_u \wedge \mathbf{x}_v \neq \mathbf{0}$ because $g'(u) \neq 0$. Since normal vector $\mathbf{x}_u \wedge \mathbf{x}_v$ is parallel to

$$(-g'(u) \cos v, -g'(u) \sin v, f'(u)),$$

the normal line at $\mathbf{x}(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$ is given by

$$\frac{x - f(u) \cos v}{-g'(u) \cos v} = \frac{y - f(u) \sin v}{-g'(u) \sin v} = \frac{z - g(u)}{f'(u)}.$$

This line passes through $\left(0, 0, g(u) + \frac{f(u)}{g'(u)}f'(u)\right)$, so all the normals to the surface all pass through the z axis.

12. We denote

$$S_1 = \{(x, y, z) \mid x^2 + y^2 + z^2 = ax\},$$
$$S_2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = by\},$$
$$S_3 = \{(x, y, z) \mid x^2 + y^2 + z^2 = cz\}.$$

Define $f_1(x, y, z) = x^2 + y^2 + z^2 - ax$. Since $d(f_1)_{\mathbf{q}} = (2x - a, 2y, 2z) = (0, 0, 0)$ implies $(x, y, z) = (a/2, 0, 0)$, the critical value of f_1 is $f_1(a/2, 0, 0) = -a^2/4 \neq 0$. Therefore $S_1 = f_1^{-1}(0)$ is a regular surface by Prop. 2, Sec. 2-2. Similarly, S_2 and S_3 are also regular surfaces.

Now we show that they all intersect orthogonally. Consider the intersections of S_1 and S_2, that is, $x^2 + y^2 + z^2 = ax = by$. Denote the unit normal vector of S_1 and S_2 at (x, y, z) as $\mathbf{N}_1(x, y, z)$ and $\mathbf{N}_2(x, y, z)$, respectively. By Exercise 1, the unit normal vectors are given by

$$\mathbf{N}_1(x, y, z) = \frac{(2x - a, 2y, 2z)}{\sqrt{(2x - a)^2 + (2y)^2 + (2z)^2}}, \quad \mathbf{N}_2(x, y, z) = \frac{(2x, 2y - b, 2z)}{\sqrt{(2x)^2 + (2y - b)^2 + (2z)^2}}.$$

Since

$$(2x - a, 2y, 2z) \cdot (2x, 2y - b, 2z) = 4(x^2 + y^2 + z^2) - 2(ax + by) = 0$$

for every $(x, y, z) \in S_1 \cap S_2$, we obtain $\mathbf{N}_1(x, y, z) \cdot \mathbf{N}_2(x, y, z) = 0$ and that S_1 and S_2 intersect orthogonally. Similarly, we can show that S_2 and S_3, S_3 and S_1 intersect orthogonally.

13. **a.** Given $\mathbf{w} \in T_{\mathbf{p}}(S)$, we can find parametrization $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = \mathbf{p}$, $\alpha'(0) = \mathbf{w}$.

$$df_{\mathbf{p}}(\mathbf{w}) = \frac{d}{dt} f(\alpha(t)) \Big|_{t=0} = \frac{d}{dt} |\alpha(t) - \mathbf{p}_0| \Big|_{t=0} = \frac{d}{dt} \sqrt{(\alpha(t) - \mathbf{p}_0) \cdot (\alpha(t) - \mathbf{p}_0)} \Big|_{t=0}$$

$$= \frac{2(\alpha(t) - \mathbf{p}_0) \cdot \alpha'(t)}{2\sqrt{(\alpha(t) - \mathbf{p}_0) \cdot (\alpha(t) - \mathbf{p}_0)}} \Big|_{t=0} = \frac{(\alpha(t) - \mathbf{p}_0) \cdot \alpha'(t)}{(\alpha(t) - \mathbf{p}_0)} \Big|_{t=0}$$

$$= \frac{(\mathbf{p} - \mathbf{p}_0) \cdot \mathbf{w}}{|\mathbf{p} - \mathbf{p}_0|}.$$

Therefore

$$df_{\mathbf{p}} = 0 \iff (\mathbf{p} - \mathbf{p}_0) \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in T_{\mathbf{p}}(S) \iff T_{\mathbf{p}}(S) \perp (\mathbf{p} - \mathbf{p}_0).$$

b. By Example 1, $dh_{\mathbf{p}}(\mathbf{w}) = \mathbf{w} \cdot \mathbf{v}$. Therefore
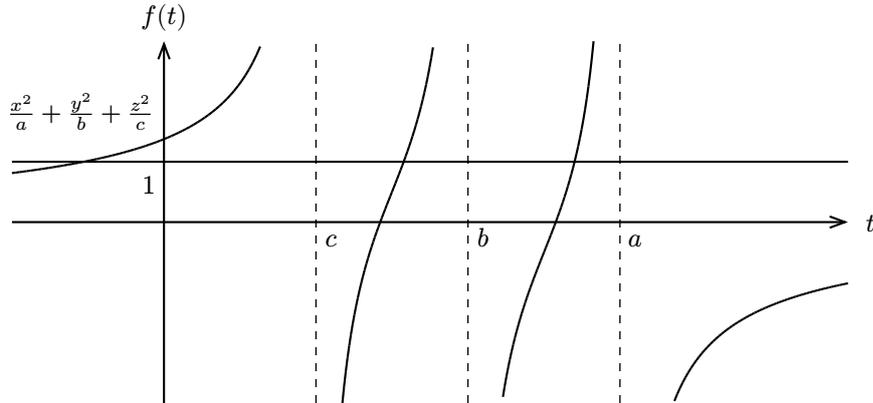
$$dh_{\mathbf{p}} = 0 \iff \mathbf{w} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{w} \in T_{\mathbf{p}}(S) \iff T_{\mathbf{p}}(S) \perp \mathbf{v}.$$

14. **a.** The graph of the function $f(t)$ is as follows. Therefore the equation $f(t) = 1$ has three distinct real roots.

b. Define

$$f_k(x, y, z) = \frac{x^2}{a - t_k} + \frac{y^2}{b - t_k} + \frac{z^2}{c - t_k} - 1$$

for $k = 1, 2, 3$. Then $d(f_k)_{\mathbf{q}} = \left(\frac{2x}{a-t}, \frac{2y}{b-t}, \frac{2z}{c-t}\right) = (0,0,0)$ implies $(x, y, z) = (0, 0, 0)$, therefore the singular value of f_k is $f_k(0, 0, 0) = -1$. Hence the sets given by $S_k = f_k^{-1}(0)$ are regular surfaces by Prop. 2, Sec. 2-2. Note that these surfaces pass through $\mathbf{p} = (x, y, z)$.

Now $\mathbf{N}_k(x, y, z)$ denote the unit normal vector of S_k at (x, y, z). By Exercise 1,

$$\mathbf{N}_k(x, y, z) \parallel \left(\frac{2x}{a - t_k}, \frac{2y}{b - t_k}, \frac{2z}{c - t_k}\right).$$

Now given distinct $k, l \in \{1, 2, 3\}$, we have

$$\left(\frac{2x}{a - t_k}, \frac{2y}{b - t_k}, \frac{2z}{c - t_k}\right) \cdot \left(\frac{2x}{a - t_l}, \frac{2y}{b - t_l}, \frac{2z}{c - t_l}\right)$$

$$= 4\left(\frac{x^2}{(a - t_k)(a - t_l)} + \frac{y^2}{(b - t_k)(b - t_l)} + \frac{z^2}{(c - t_k)(c - t_l)}\right)$$

$$= \frac{4}{t_k - t_l}\left(\left(\frac{x^2}{a - t_k} + \frac{y^2}{b - t_k} + \frac{z^2}{c - t_k} - 1\right) - \left(\frac{x^2}{a - t_l} + \frac{y^2}{b - t_l} + \frac{z^2}{c - t_l} - 1\right)\right)$$

$$= 0.$$

Therefore $\mathbf{N}_k(x, y, z) \cdot \mathbf{N}_l(x, y, z) = 0$ for every $(x, y, z) \in S_k \cap S_l$, that is, they are pairwise orthogonal.

15. Suppose that all normals to a connected surface S pass through the fixed point $\mathbf{p}_0$. Define $f : S \to \mathbb{R}$ as $f(\mathbf{p}) = |\mathbf{p} - \mathbf{p}_0|^2$. By Exercise 7, the differential of f is $df_{\mathbf{p}}(\mathbf{w}) = 2\mathbf{w} \cdot (\mathbf{p} - \mathbf{p}_0)$. Since the normal at $\mathbf{p}$ passes through $\mathbf{p}_0$, we have $\mathbf{w} \cdot (\mathbf{p} - \mathbf{p}_0)$ for every $\mathbf{w} \in T_{\mathbf{p}}(S)$. Therefore $df_{\mathbf{p}} = 0$ for all $\mathbf{p} \in S$. Since S is a connected surface, f is constant on S by Exercise 21, which means that the surface is contained in a sphere.

16. For parametrizations $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ and $\overline{\mathbf{x}} : \overline{U} \subset \mathbb{R}^2 \to S$, let $W = \mathbf{x}(U) \cap \overline{\mathbf{x}}(\overline{U})$. Then the change of coordinates is given by

$$\overline{\mathbf{x}}^{-1} \circ \mathbf{x} : \mathbf{x}^{-1}(W) \to \overline{\mathbf{x}}^{-1}(W), \quad (\overline{\mathbf{x}}^{-1} \circ \mathbf{x})(u, v) = (\overline{u}(u, v), \overline{v}(u, v)).$$

Hence $\mathbf{x}(u, v) = \overline{\mathbf{x}}(\overline{u}(u, v), \overline{v}(u, v))$. Differentiating both sides with respect to u and v, respectively, then

$$\mathbf{x}_u = \overline{\mathbf{x}}_{\overline{u}}\frac{\partial \overline{u}}{\partial u} + \overline{\mathbf{x}}_{\overline{v}}\frac{\partial \overline{v}}{\partial u}, \quad \mathbf{x}_v = \overline{\mathbf{x}}_{\overline{u}}\frac{\partial \overline{u}}{\partial v} + \overline{\mathbf{x}}_{\overline{v}}\frac{\partial \overline{v}}{\partial v}.$$

Since

$$\mathbf{w} = \alpha_1 \mathbf{x}_u + \alpha_2 \mathbf{x}_v = \alpha_1\left(\overline{\mathbf{x}}_{\overline{u}}\frac{\partial \overline{u}}{\partial u} + \overline{\mathbf{x}}_{\overline{v}}\frac{\partial \overline{v}}{\partial u}\right) + \alpha_2\left(\overline{\mathbf{x}}_{\overline{u}}\frac{\partial \overline{u}}{\partial v} + \overline{\mathbf{x}}_{\overline{v}}\frac{\partial \overline{v}}{\partial v}\right)$$

$$= \left(\alpha_1\frac{\partial \overline{u}}{\partial u} + \alpha_2\frac{\partial \overline{u}}{\partial v}\right)\overline{\mathbf{x}}_{\overline{u}} + \left(\alpha_1\frac{\partial \overline{v}}{\partial u} + \alpha_2\frac{\partial \overline{v}}{\partial v}\right)\overline{\mathbf{x}}_{\overline{v}} = \beta_1\overline{\mathbf{x}}_{\overline{u}} + \beta_2\overline{\mathbf{x}}_{\overline{v}},$$

we conclude that the coordinates of $\mathbf{w}$ are related by

$$\beta_1 = \alpha_1\frac{\partial \overline{u}}{\partial u} + \alpha_2\frac{\partial \overline{u}}{\partial v}, \quad \beta_2 = \alpha_1\frac{\partial \overline{v}}{\partial u} + \alpha_2\frac{\partial \overline{v}}{\partial v}.$$

17. In the neighborhood of $\mathbf{p} \in S_1 \cap S_2$, S_1 can be expressed by one of the equations $z = a(x, y), y = b(x, z), x = c(y, z)$ by Prop. 3, Sec 2-2. We can assume, by renaming the axes if necessary, that S_1 can be expressed by the equation $z = a(x, y)$. Then we can define $f(x, y, z) = z - a(x, y)$ and express S_1 by $f^{-1}(0)$ in the neighborhood of $\mathbf{p}$. Note that $df_{\mathbf{q}} = (*, *, 1) \neq (0, 0, 0)$, so 0 is a regular value of f. Similarly, there exists the function g such that $g(x, y, z) = 0$ expresses S_2 in the neighborhood of $\mathbf{p}$.

Now define $\mathbf{F} : U \subset \mathbb{R}^3 \to \mathbb{R}^2$ as $\mathbf{F}(x, y, z) = (f(x, y, z), g(x, y, z))$ in the neighborhood U of $\mathbf{p}$. Note that $S_1 \cap S_2$ is given as the inverse image of $(0, 0)$ under $\mathbf{F}$ in the neighborhood of $\mathbf{p}$. Since S_1 intersects S_2 transversally, the normal vectors (f_x, f_y, f_z) and (g_x, g_y, g_z) are linearly independent. Therefore the differential of F,

$$d\mathbf{F_q} = \begin{pmatrix} f_x & f_y & f_x \\ g_x & g_y & g_x \end{pmatrix},$$

is surjective for every $\mathbf{q} \in U \cap (S_1 \cap S_2)$. Hence $\mathbf{0} = (0,0)$ is a regular value of $\mathbf{F}$. Since $S_1 \cap S_2$ can be expressed locally by $\mathbf{F}^{-1}(\mathbf{0})$, which is a regular curve by Exercise 17(b) of Sec. 2-2, we can conclude that $S_1 \cap S_2$ is a regular curve.

18. Suppose that $T_{\mathbf{p}}(S) \neq P$. Since $T_{\mathbf{p}}(P) = P$, we have $T_{\mathbf{p}}(S) \neq T_{\mathbf{p}}(P)$. Then Exercise 17 shows that $S \cap P = \{\mathbf{p}\}$ is a regular curve, which is a contradiction; therefore, the plane P coincides with the tangent plane of S at $\mathbf{p}$.

19. Choose arbitrary $\mathbf{p} \in P \cap S$. Let $\mathbf{v}$ be an unit normal vector of P, pointing the side where S exists. Now define $f : S \to \mathbb{R}$ as $f(\mathbf{x}) = (\mathbf{x} - \mathbf{p}) \cdot \mathbf{v}$. For any $\mathbf{w} \in T_{\mathbf{p}}(S)$, we can find parametrized curve $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = \mathbf{p}$ and $\alpha'(0) = \mathbf{w}$. Since $f(\alpha(t)) = (\alpha(t) - \mathbf{p}) \cdot \mathbf{v} \geq 0$ for all $t \in (-\varepsilon, \varepsilon)$ and $f(\alpha(0)) = \mathbf{0} \cdot \mathbf{v} = 0$, we have $\frac{d}{dt} f(\alpha(t))|_{t=0} = 0$, that is, $df_{\mathbf{p}}(\mathbf{w}) = 0$ for every $\mathbf{w} \in T_{\mathbf{p}}(S)$. Note that $df_{\mathbf{p}}(\mathbf{w}) = \mathbf{w} \cdot \mathbf{v}$ by Example 1, so $\mathbf{w} \cdot \mathbf{v} = 0$ for every $\mathbf{w} \in T_{\mathbf{p}}(S)$. Therefore $T_{\mathbf{p}}(S) = P$.

20. Since the ellipsoid S is the inverse image of $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$ at regular value 0, it is a regular surface. We first show that if $(x, y, z) \neq \mathbf{0}$ is the perpendicular projection of the center $\mathbf{0}$ onto its tangent plane, then it satisfies the equation $(x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2$. By Exercise 1, the equation of the tangent plane at (x_0, y_0, z_0) is

$$\frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) + \frac{2z_0}{c^2}(z - z_0) = 0 \implies \frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1.$$

The line through $\mathbf{0}$ and perpendicular to the tangent plane is given by

$$\frac{xa^2}{x_0} = \frac{yb^2}{y_0} = \frac{zc^2}{z_0}.$$

From the last expression, we obtain

$$\frac{x^2a^2}{xx_0} = \frac{y^2b^2}{yy_0} = \frac{z^2c^2}{zz_0} = \frac{a^2x^2 + b^2y^2 + c^2z^2}{xx_0 + yy_0 + zz_0}.$$

From the same expression, and taking into account the equation of the ellipsoid, we obtain

$$\frac{xx_0}{x_0^2/a^2} = \frac{yy_0}{y_0^2/b^2} = \frac{zz_0}{z_0^2/c^2} = \frac{xx_0 + yy_0 + zz_0}{1}.$$

Again from the same expression and using the equation of the tangent plane, we obtain

$$\frac{x^2}{(x_0 x)/a^2} = \frac{y^2}{(y_0 y)/b^2} = \frac{z^2}{(z_0 z)/c^2} = \frac{x^2 + y^2 + z^2}{1}.$$

The right-hand sides of the three last equations are therefore equal, and hence the asserted equation. Conversely, suppose that $(x, y, z) \neq \mathbf{0}$ satisfies the given equation. From the last equation, we can calculate (x_0, y_0, z_0) from (x, y, z), that is,

$$(x_0, y_0, z_0) = \left(\frac{xa^2}{x^2 + y^2 + z^2}, \frac{yb^2}{x^2 + y^2 + z^2}, \frac{zc^2}{x^2 + y^2 + z^2} \right).$$

Then we can see that (x, y, z) is the perpendicular projection of the center $\mathbf{0}$ onto the tangent plane at (x_0, y_0, z_0). Therefore the trace of the perpendicular projections constitutes a surface given by

$$T = \left\{ (x, y, z) \in \mathbb{R}^3 \mid (x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2 \right\} - \{(0,0,0)\}.$$

Now we show that T is a regular surface. Define $g : \mathbb{R}^3 - \{0\} \to \mathbb{R}$ as

$$g(x, y, z) = (x^2 + y^2 + z^2)^2 - (a^2x^2 + b^2y^2 + c^2z^2).$$

Since

$$dg_{\mathbf{q}} = \left(4x\left(x^2 + y^2 + z^2 - \frac{a^2}{2}\right), 4y\left(x^2 + y^2 + z^2 - \frac{b^2}{2}\right), 4z\left(x^2 + y^2 + z^2 - \frac{c^2}{2}\right)\right),$$

the equation $dg_{\mathbf{q}} = (0,0,0)$ implies

$$\left(x = 0 \vee x^2 + y^2 + z^2 = \frac{a^2}{2}\right) \wedge \left(y = 0 \vee x^2 + y^2 + z^2 = \frac{b^2}{2}\right) \wedge \left(z = 0 \vee x^2 + y^2 + z^2 = \frac{c^2}{2}\right).$$

There are several cases depending on the values of a, b, c. If they are all distinct, then at least 2 variables must be zero. For example, $x^2 + y^2 + z^2 = a^2/2$ and $y = z = 0$ then $g(x,y,z) = (x^2)^2 - a^2 x^2 = (a^2/2)^2 - a^2(a^2/2) = -a^4/4 \neq 0$.

Next, suppose that two values are equal, e.g., $a = b \neq c$. This allows for the case $x^2 + y^2 + z^2 = a^2/2 = b^2/2$ and $z = 0$, and $g(x,y,z) = (x^2 + y^2)^2 - (a^2 x^2 + b^2 y^2) = (a^2/2)^2 - a^2(a^2/2) = -a^2/4 \neq 0$.

Finally, if $a = b = c$, then this allows for the case $x^2 + y^2 + z^2 = a^2/2 = b^2/2 = c^2/2$ and $g(x,y,z) = (x^2 + y^2 + z^2)^2 - (a^2 x^2 + b^2 y^2 + c^2 z^2) = (a^2/2)^2 - a^2(a^2/2) = -a^2/4 \neq 0$.

In all cases, we can see that 0 is a regular value of g, hence $S = g^{-1}(0)$ is a regular surface by Prop. 2, Sec. 2-2.

21. Let $\mathbf{p} \in S$. Since S is a regular surface, there exists a circular neighborhood V in $\mathbb{R}^3$ and a parametrization $\mathbf{x} : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$. Any point $\mathbf{q} \in V \cap S$ can be joined to $\mathbf{p}$ by the segment $\beta : [0,1] \to V \cap S$, where $\beta(t) = \mathbf{x}(t\mathbf{x}^{-1}(\mathbf{p}) + (1-t)\mathbf{x}^{-1}(\mathbf{q}))$. Since $V \cap S$ is open, we can extend β to $(0 - \varepsilon, 1 + \varepsilon)$. Now, $f \circ \beta : (0 - \varepsilon, 1 + \varepsilon) \to \mathbb{R}$ is a function defined in an open interval, and $d(f \circ \beta)_t = df_{\beta(t)} \circ d\beta_t = 0$, since $df_{\mathbf{p}} \equiv 0$. Thus, $\frac{d}{dt}(f \circ \beta) = 0$ for all $t \in (0 - \varepsilon, 1 + \varepsilon)$, and hence $(f \circ \beta) = $ const.. This means that $f(\mathbf{p}) = f(\beta(0)) = f(\beta(1)) = f(\mathbf{q})$; that is, f is constant on $V \cap S$.

Let $\mathbf{r}$ be an arbitrary point of S. Since S is connected, there exists a continuous curve $\alpha : [a,b] \to \mathbb{R}$, with $\alpha(a) = \mathbf{p}, \alpha(b) = \mathbf{r}$. The function $\alpha : [a,b] \to \mathbb{R}$ is continuous in $[a,b]$. By the first part of the proof, for each $t \in [a,b]$, there exists an interval I_t, open in $[a,b]$, such that $f \circ \alpha$ is constant on I_t. Since $\bigcup_t I_t = [a,b]$, we can apply the Heine-Borel theorem. Thus, we can choose a finite number $I_1, ..., I_k$ of the intervals I_t so that $\bigcup_i I_i = [a,b], i = 1, ..., k$. We can assume, by renumbering the intervals, if necessary, that two consecutive intervals overlap. Thus, $f \circ \alpha$ is constant in the union of two consecutive intervals. It follows that f is constant on $[a,b]$; that is,

$$f(\mathbf{p}) = f(\alpha(a)) = f(\alpha(b)) = f(\mathbf{r}).$$

Since $\mathbf{r}$ is arbitrary, f is constant on S.

22. Let r be a fixed line which is met by the normals of S, and let $\mathbf{p} \in S$. The plane P_1, which contains $\mathbf{p}$ and r, contains all the normals to S at the points of $P_1 \cap S$. Consider a plane P_2 passing through $\mathbf{p}$ and perpendicular to r. Since the normal through $\mathbf{p}$ meets r, P_2 is transversal to $T_{\mathbf{p}}(S)$. Hence, $P_2 \cap S$ is a regular plane curve C in a neighborhood of $\mathbf{p}$, by Exercise 17. Furthermore, $P_1 \cap P_2$ is perpendicular to $T_{\mathbf{p}}(S) \cap P_2$. Hence $P_1 \cap P_2$ is normal to C. It follows that the normal at $\mathbf{p}$ pass through a point $\mathbf{q} = r \cap P_2$. Since $\mathbf{q}$ is determined by the plane P_2, we can conclude that the normals s of C all pass through a fixed point $\mathbf{q} = r \cap P_2$. Hence C is contained in a circle by Exercise 4 of Sec 1-5. Finally, P_1 is also transversal to $T_{\mathbf{p}}(S)$, so $P_1 \cap S$ is a regular plane curve. Then S, in the neighborhood of $\mathbf{p}$, can be obtained by rotating a regular plane curve $P_1 \cap S$. Thus every $\mathbf{p} \in S$ has a neighborhood contained in some surface of revolution with axis r.

Note We proved that S is locally piece of a surface of revolution at every point $\mathbf{p} \in S$. However, we can't extend this property to global. For example, there is a counterexample such that all normal lines to a connected regular surface S meets a fixed straight line, but S is not a piece of a surface of revolution. Define $\mathbf{x} : U \to R^3$ in the open set $U = (0, 2\pi) \times (0,1) \cup ((0, \pi/2) \cup (\pi, 3\pi/2) \times [1,3)$ as

$$\mathbf{x}(\theta, z) = (r(\theta, z) \cos\theta, r(\theta, z) \sin\theta, z),$$

where

$$r(\theta, z) = \begin{cases} 1 & (0 < z < 1) \\ r^*(z) & (1 \le z < 2, \ 0 < \theta < \pi/2) \\ 1 & (1 \le z < 2, \ \pi < \theta < 3\pi/2) \\ 1/2 & (2 \le z < 3, \ 0 < \theta < \pi/2) \\ 1 & (2 \le z < 3, \ \pi < \theta < 3\pi/2) \end{cases}$$

and a bump function $r^*(z)$ is a differentiable function satisfying $r^*(1) = 1$ and $r^*(2) = 1/2$, e.g.,

$$\psi(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \le 0 \end{cases}, \quad \sigma(t) = \frac{\psi(1-t)}{\psi(1-t) + \psi(t)}, \quad r^*(z) = \frac{1}{2} + \frac{1}{2}\sigma(z-1).$$

Then this surface $S = \mathbf{x}(U)$ satisfies the condition that all normals meet a fixed straight line (z axis) because it is locally a piece of surface of revolution at every point in S, but it is not a piece of surface of revolution.
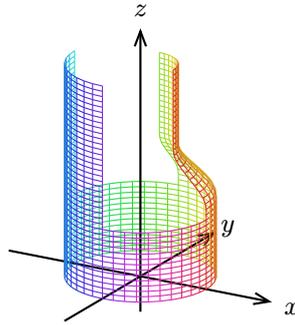
23. Recall that $\mathbf{F}(\mathbf{p}) = (\pi_N^{-1} \circ P \circ \pi_N)(\mathbf{p})$ for $\mathbf{p} \in \mathbb{S}^2 - \{N\}$, and $\mathbf{F}(N) = N$. We assert that there is only a finite number of critical points in $\mathbb{S}^2 - \{N\}$. Since $\pi_N^{-1} : \mathbb{R}^2 \to \mathbb{S}^2 - \{N\}$ is defined by

$$\pi_N^{-1}(u, v) = \left(\frac{4u}{u^2 + v^2 + 4}, \frac{4v}{u^2 + v^2 + 4}, \frac{u^2 + v^2 - 4}{u^2 + v^2 + 4} \right),$$

we obtain

$$d\left(\pi_N^{-1}\right)_{\mathbf{q}} = \begin{pmatrix} \frac{-4u^2+4v^2+16}{(u^2+v^2+4)^2} & \frac{-8uv}{(u^2+v^2+4)^2} \\ \frac{-8uv}{(u^2+v^2+4)^2} & \frac{4u^2-4v^2+16}{(u^2+v^2+4)^2} \\ \frac{16u}{(u^2+v^2+4)^2} & \frac{16v}{(u^2+v^2+4)^2} \end{pmatrix}.$$

Therefore there is no $\mathbf{q} \in \mathbb{R}^2$ such that $d\left(\pi_N^{-1}\right)_{\mathbf{q}} = \mathbf{0}$. Similarly, there is no point $\mathbf{p} \in \mathbb{S}^2 - \{N\}$ such that

$$d\left(\pi_N^{-1}\right)_{\mathbf{p}} = \left(d\left(\pi_N^{-1}\right)_{\pi_N(\mathbf{p})} \right)^{-1} = \mathbf{0}.$$

We now consider the complex polynomial $P(\zeta) = a_0 \zeta^n + a_1 \zeta^{n-1} + \cdots + a_n$. If we consider P as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$, then for the differential $dP_{\mathbf{q}}$, it follows that $\det(dP_{\mathbf{q}}) = |P'(\zeta)|^2$. (See Proposition 2.3 in Stein's Complex Analysis.) Therefore $dP_{\mathbf{q}} = 0$ only if $P'(\zeta) = 0$. Since P is a polynomial of degree n, its derivative P' has degree $n-1$, and thus there are at most $n-1$ such points. We conclude that $\mathbf{F}$ has at most n critical points, including the potential critical point at N.

24. We first show that if $\varphi : S_1 \to S_2$ and $\psi : S_2 \to S_3$ are differentiable maps, then $\psi \circ \varphi : S_2 \to S_3$ is differentiable. Given $\mathbf{p} \in S_1$, let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be a parametrizations of S_1, S_2, S_3, at $\mathbf{p}, \varphi(\mathbf{p}), \psi(\varphi(\mathbf{p}))$, respectively. Then $\mathbf{x}_2^{-1} \circ \varphi \circ \mathbf{x}_1$ is differentiable at $\mathbf{x}_1^{-1}(\mathbf{p})$ and $\mathbf{x}_3^{-1} \circ \psi \circ \mathbf{x}_2$ is differentiable at $\mathbf{x}_2^{-1}(\varphi(\mathbf{p}))$. Therefore

$$\mathbf{x}_3^{-1} \circ (\psi \circ \varphi) \circ \mathbf{x}_1 = \left(\mathbf{x}_3^{-1} \circ \psi \circ \mathbf{x}_2 \right) \circ \left(\mathbf{x}_2^{-1} \circ \varphi \circ \mathbf{x}_1 \right)$$

is differentiable at $\mathbf{x}_1^{-1}(\mathbf{p})$. Since $\mathbf{p} \in S_1$ is arbitrary, $\psi \circ \varphi$ is a differentiable map.

Now fix any $\mathbf{w} \in T_{\mathbf{p}}(S_1)$, and find a parametrized curve $\alpha : (-\varepsilon, \varepsilon) \to S_1$ with $\alpha(0) = \mathbf{p}$ and $\alpha'(0) = \mathbf{w}$. By definition, the map $d(\psi \circ \varphi)_{\mathbf{p}} : T_{\mathbf{p}}(S_1) \to T_{(\psi \circ \varphi)(\mathbf{p})}(S_3)$ is given by

$$d(\psi \circ \varphi)_{\mathbf{p}}(\mathbf{w}) = \gamma'(0)$$

where $\gamma = \psi \circ \varphi \circ \alpha$. Let $\beta = \varphi \circ \alpha$, then we obtain

$$d\varphi_{\mathbf{p}}(\mathbf{w}) = (\varphi \circ \alpha)'(0) = \beta'(0),$$

$$d\psi_{\varphi(\mathbf{p})}(\beta'(0)) = (\psi \circ \beta)'(0) = \gamma'(0).$$

Therefore

$$\left(d\psi_{\varphi(\mathbf{p})} \circ d\varphi_{\mathbf{p}}\right)(\mathbf{w}) = \gamma'(0).$$

Since $\mathbf{w} \in T_{\mathbf{p}}(S_1)$ is arbitrary, we conclude that $d(\psi \circ \varphi)_{\mathbf{p}} = d\psi_{\varphi(\mathbf{p})} \circ d\varphi_{\mathbf{p}}$.

25. Note that the tangent vector at $\mathbf{p}$ of regular curve C, which is parametrized by $\alpha : I \to V \cap C$ at $\mathbf{p} = \alpha(t_0)$, is given as $\alpha'(t_0)$.

We first show that $\varphi(C_1)$ and $\varphi(C_2)$ are regular curves. Since the curves C_1 and C_2 are regular, there are parametrizations $\alpha_1 : I_1 \to V_1 \cap C_1$, where $d(\alpha_1)_t$ is one-to-one for each $t \in I_1$, and $\alpha_2 : I_2 \to V_2 \cap C_2$, where $d(\alpha_2)_t$ is one-to-one for each $t \in I_2$. Fix arbitrary $\varphi(\mathbf{p}) \in \varphi(C_1)$, and we will show that the map $\varphi \circ \alpha_1 : I_1 \to \varphi(V_1 \cap C_1) \subset V_1' \cap \varphi(C_1)$ is a parametrization for $\varphi(C_1)$ at $\varphi(\mathbf{p})$. Differentiating both sides of $\varphi^{-1} \circ \varphi = \mathrm{Id}_S$ yields $d(\varphi^{-1})_{\varphi(\mathbf{p})} \circ d\varphi_{\mathbf{p}} = d(\mathrm{Id}_S)_{\mathbf{p}} = \mathrm{Id}_{T_{\mathbf{p}}(S)}$. Since $\mathrm{Id}_{T_{\mathbf{p}}(S)}$ is one-to-one, $d\varphi_{\mathbf{p}}$ is one-to-one for every $\mathbf{p} \in S$. therefore $d(\varphi \circ \alpha_1)_t = d\varphi_{\alpha_1(t)} \circ d(\alpha_1)_t$ is a composition of injective mappings, which is also one-to-one. Therefore $\varphi(C_1)$ is a regular curve. Similarly, $\varphi(C_2)$ is a regular curve.

Now suppose that regular curves C_1 and C_2 are tangent at a point $\mathbf{p} = \alpha_1(t_1) = \alpha_2(t_2) \in S$, that is, $\alpha_1'(t_1) \parallel \alpha_2'(t_2)$. Then we have $c_1 \alpha_1'(t_1) + c_2 \alpha_2'(t_2) = \mathbf{0}$ for $c_1, c_2 \in \mathbb{R}$, $c_1^2 + c_2^2 \neq 0$. Since

$$(\varphi \circ \alpha_1)'(t_1) = d\varphi_{\alpha_1(t_1)}(\alpha_1'(t_1)) = d\varphi_{\mathbf{p}}(\alpha_1'(t_1)),$$

$$(\varphi \circ \alpha_2)'(t_2) = d\varphi_{\alpha_2(t_2)}(\alpha_2'(t_2)) = d\varphi_{\mathbf{p}}(\alpha_2'(t_2)),$$

and $d\varphi_{\mathbf{p}}$ is linear, we obtain

$$c_1 d\varphi_{\mathbf{p}}(\alpha_1'(t_1)) + c_2 d\varphi_{\mathbf{p}}(\alpha_2'(t_2)) = d\varphi_{\mathbf{p}}(c_1 \alpha_1'(t_1) + c_2 \alpha_2'(t_2)) = d\varphi_{\mathbf{p}}(\mathbf{0}) = \mathbf{0}.$$

Hence $\varphi(C_1)$ and $\varphi(C_2)$ are regular curves which are tangent at $\varphi(\mathbf{p})$.

26. Let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ be a parametrization of S in $\mathbf{p}$, and write $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$. Take any orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ of $T_{\mathbf{p}}(S)$ and let $\mathbf{v}_3 = \pm(\mathbf{x}_u \wedge \mathbf{x}_v)/|\mathbf{x}_u \wedge \mathbf{x}_v|$, where the sign of $\mathbf{v}_3$ is chosen so that $\det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = 1$. Then we have $\mathbf{v}_1 \wedge \mathbf{v}_2 = \mathbf{v}_3$. For the point $\mathbf{x} \in \mathbb{R}^3$ in the original coordinate system, we define the new coordinates $\overline{\mathbf{x}} = (\overline{x}, \overline{y}, \overline{z})$ as

$$\overline{x} = (\mathbf{x} - \mathbf{p}) \cdot \mathbf{v}_1, \quad \overline{y} = (\mathbf{x} - \mathbf{p}) \cdot \mathbf{v}_2, \quad \overline{z} = (\mathbf{x} - \mathbf{p}) \cdot \mathbf{v}_3, \quad \overline{\mathbf{x}} = (\overline{x}, \overline{y}, \overline{z}).$$

Now we represent the neighborhood of $\mathbf{p}$ in S in the form $\overline{z} = \overline{f}(\overline{x}, \overline{y})$. Define $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ as

$$\pi(x, y, z) = (\overline{x}(x, y, z), \overline{y}(x, y, z))$$

and consider the map $\pi \circ \mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^2$. Since

$$\frac{\partial(\overline{x}, \overline{y})}{\partial(u, v)} = \left| \begin{pmatrix} \partial \overline{x}/\partial u & \partial \overline{x}/\partial v \\ \partial \overline{y}/\partial u & \partial \overline{y}/\partial v \end{pmatrix} \right| = \left| \begin{pmatrix} \mathbf{x}_u \cdot \mathbf{v}_1 & \mathbf{x}_v \cdot \mathbf{v}_1 \\ \mathbf{x}_u \cdot \mathbf{v}_2 & \mathbf{x}_v \cdot \mathbf{v}_2 \end{pmatrix} \right|$$

$$= (\mathbf{x}_u \cdot \mathbf{v}_1)(\mathbf{x}_v \cdot \mathbf{v}_2) - (\mathbf{x}_u \cdot \mathbf{v}_2)(\mathbf{x}_v \cdot \mathbf{v}_1)$$

$$= (\mathbf{x}_u \wedge \mathbf{x}_v) \cdot (\mathbf{v}_1 \wedge \mathbf{v}_2) = (\mathbf{x}_u \wedge \mathbf{x}_v) \cdot \mathbf{v}_3$$

$$= |\mathbf{x}_u \wedge \mathbf{x}_v| \neq 0$$

at $\mathbf{q} = \mathbf{x}^{-1}(\mathbf{p})$, we can apply the inverse function theorem to guarantee the existence of neighborhoods V_1 of $\mathbf{q}$, V_2 of $(\pi \circ \mathbf{x})(\mathbf{q})$ such that $\pi \circ \mathbf{x}$ maps V_1 diffeomorphically onto V_2. It follows that π restricted to $\mathbf{x}(V_1) = V$ is one-to-one and that there is a differentiable inverse $(\pi \circ \mathbf{x})^{-1} : V_2 \to V_1$. Since $\mathbf{x}$ is a homeomorphism, V is a neighborhood of $\mathbf{p}$ in S. We compose the map $(\pi \circ \mathbf{x})^{-1} : (\overline{x}, \overline{y}) \mapsto (u(\overline{x}, \overline{y}), v(\overline{x}, \overline{y}))$ and the map $(u, v) \mapsto \overline{z}(\mathbf{x}(u, v)) = (\mathbf{x}(u, v) - \mathbf{p}) \cdot \mathbf{v}_3$, then we find that V is the graph of the differentiable function

$$\overline{z} = \overline{z}(u(\overline{x}, \overline{y}), v(\overline{x}, \overline{y})) = \overline{f}(\overline{x}, \overline{y}).$$

Now we show that $\overline{f}(0,0) = \overline{f}_{\overline{x}}(0,0) = \overline{f}_{\overline{y}}(0,0) = 0$. Since the surface S passes the point $\mathbf{p}$, which has the coordinates $(\overline{x}, \overline{y}, \overline{z}) = (0, 0, 0)$, we have that $\overline{f}(0,0) = 0$. Moreover,

$$\frac{\partial \overline{f}}{\partial \overline{x}} = \frac{\partial \overline{z}}{\partial u}\frac{\partial u}{\partial \overline{x}} + \frac{\partial \overline{z}}{\partial v}\frac{\partial v}{\partial \overline{x}} = (\mathbf{x}_u \cdot \mathbf{v}_3)\frac{\partial u}{\partial \overline{x}} + (\mathbf{x}_v \cdot \mathbf{v}_3)\frac{\partial v}{\partial \overline{x}} = 0$$

at $\mathbf{p}$, because $\mathbf{x}_u, \mathbf{x}_v \in T_{\mathbf{p}}(S)$ and $\mathbf{v}_3$ is the unit normal vector of $T_{\mathbf{p}}(S)$. Thus we have $\overline{f}_{\overline{x}}(0,0) = 0$. Similarly, we obtain $\overline{f}_{\overline{y}}(0,0) = 0$.

27. a. Let $\mathbf{x}(u,v)$ be the parametrization of S at $\mathbf{p}$, where $\mathbf{x}(u_0, v_0) = \mathbf{p}$. We assert that

$$\overline{\mathbf{x}}(u,v) = \mathbf{p} + \mathbf{x}_u(u_0, v_0)(u - u_0) + \mathbf{x}_v(u_0, v_0)(v - v_0)$$

is a parametrization for $T_{\mathbf{p}}(S)$ at $\mathbf{p}$. (i) It is easy to see that $\overline{\mathbf{x}}$ is differentiable. (iii) Since

$$d\overline{\mathbf{x}}_{\mathbf{q}} = \left(\mathbf{x}_u(u_0, v_0) \,\middle|\, \mathbf{x}_v(u_0, v_0) \right),$$

$d\overline{\mathbf{x}}_{\mathbf{q}}$ is one-to-one for each $\mathbf{q}$. (ii) $\overline{\mathbf{x}}$ is one-to-one because $\mathbf{x}_u(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0)$ are linearly independent. By Prop. 4 of Sec. 2-2, $\mathbf{x}^{-1}$ is continuous. Therefore $\overline{\mathbf{x}}$ is a parametrization for $T_{\mathbf{p}}(S)$ at $\mathbf{p}$. Since $\mathbf{x}_u = \overline{\mathbf{x}}_u$ and $\mathbf{x}_v = \overline{\mathbf{x}}_v$ at (u_0, v_0), we conclude that $T_{\mathbf{p}}(S)$ has contact of order ≥ 1 with the surface at $\mathbf{p}$.

b. Let $\mathbf{x}(u,v)$ be a parametrization for S at $\mathbf{p}$. By (a), there exists a parametrization $\overline{\mathbf{x}}(u,v)$ for $T_{\mathbf{p}}(S)$ at $\mathbf{p}$, which satisfies the equation $\mathbf{x}_u = \overline{\mathbf{x}}_u$ and $\mathbf{x}_v = \overline{\mathbf{x}}_v$. Since a plane P has contact of order ≥ 1 with a surface S at $\mathbf{p}$, there exists a parametrization $\tilde{\mathbf{x}}(u,v)$ for P at $\mathbf{p}$, with $\mathbf{x}_u = \tilde{\mathbf{x}}_u$ and $\mathbf{x}_v = \tilde{\mathbf{x}}_v$. Therefore we obtain $\overline{\mathbf{x}}_u = \tilde{\mathbf{x}}_u$ and $\overline{\mathbf{x}}_v = \tilde{\mathbf{x}}_v$. Observe that two planes $T_{\mathbf{p}}(S)$ and P pass through the same point $\mathbf{p}$ and normal to $\overline{\mathbf{x}}_u \wedge \overline{\mathbf{x}}_v = \tilde{\mathbf{x}}_u \wedge \tilde{\mathbf{x}}_v$. Hence $T_{\mathbf{p}}(S) = P$.

c. First suppose that two regular surfaces S and $\overline{S}$ have contact of order ≥ 1. Then there exist parametrizations $\mathbf{x}(u,v)$, $\overline{\mathbf{x}}(u,v)$ at $\mathbf{p}$ of S and $\overline{S}$, respectively, such that $\mathbf{x}_u = \overline{\mathbf{x}}_u$ and $\mathbf{x}_v = \overline{\mathbf{x}}_v$ at $\mathbf{p}$. Since both $T_{\mathbf{p}}(S)$ and $T_{\mathbf{p}}(\overline{S})$ contain the point $\mathbf{p}$ and perpendicular to $\mathbf{x}_u \wedge \overline{\mathbf{x}}_u = \mathbf{x}_v \wedge \overline{\mathbf{x}}_v$, these planes are the same.

Conversely, suppose that the regular curves S and $\overline{S}$ have a common tangent plane at $\mathbf{p}$. Let

$$\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v)), \quad \overline{\mathbf{x}}(u,v) = (\overline{x}(u,v), \overline{y}(u,v), \overline{z}(u,v))$$

be a parametrizations at $\mathbf{p}$ for S and $\overline{S}$, respectively, with $\mathbf{x}(0,0) = \overline{\mathbf{x}}(0,0) = \mathbf{p}$. Since two surfaces have a common tangent plane at $\mathbf{p}$, we have $\mathbf{x}_u \wedge \mathbf{x}_v \perp \overline{\mathbf{x}}_u \wedge \overline{\mathbf{x}}_v$. Observe that

$$\mathbf{x}_u \wedge \mathbf{x}_v = \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right).$$

Since $\mathbf{x}_u \wedge \mathbf{x}_v \neq \mathbf{0}$, one of the Jacobian determinants is not zero. Without loss of generality, we suppose that $\partial(x,y)/\partial(u,v) \neq 0$ at $\mathbf{0}$. Then we can express the surface S as the form $z = f(x,y)$ in the neighborhood of $\mathbf{p}$. Note that $\partial(\overline{x}, \overline{y})/\partial(u,v)$ is also nonzero, since $\mathbf{x}_u \wedge \mathbf{x}_v$ and $\overline{\mathbf{x}}_u \wedge \overline{\mathbf{x}}_v$ is parallel. Therefore we can also represent the surface $\overline{S}$ as the form $z = \overline{f}(x,y)$ in the neighborhood of $\mathbf{p}$. Now we define

$$\mathbf{y}(u,v) = (u, v, f(u,v)), \quad \overline{\mathbf{y}}(u,v) = \left(u, v, \overline{f}(u,v) \right)$$

be the parametrizations at $\mathbf{p}$ for S and $\overline{S}$. Since $\mathbf{y}_u = \overline{\mathbf{y}}_u = (1,0,0)$ and $\mathbf{y}_v = \overline{\mathbf{y}}_v = (0,1,0)$, we conclude that S and $\overline{S}$ have contact of order ≥ 1.

d. Let $\mathbf{x} : U \to S$ be a parametrization of S at given point $\mathbf{p}$. We shall show that $\mathbf{F} \circ \mathbf{x} : U \to \mathbf{F}(S)$ is a parametrization for $\mathbf{F}(S)$ at $\mathbf{F}(\mathbf{p})$. (i) $\mathbf{F} \circ \mathbf{x}$ is differentiable, since it is composition of two differentiable functions $\mathbf{F}$ and $\mathbf{x}$. (ii) $\mathbf{F} \circ \mathbf{x}$ is a homeomorphism, since both $\mathbf{F}$ and $\mathbf{x}$ are homeomorphisms. (iii) We show that $d(\mathbf{F} \circ \mathbf{x})_{\mathbf{q}} = d\mathbf{F}_{\mathbf{x}(\mathbf{q})} \circ d\mathbf{x}_{\mathbf{q}}$ is one-to-one for each $\mathbf{q}$. $d\mathbf{x}_{\mathbf{q}}$ is one-to-one since $\mathbf{x}$ is a parametrization. From the equation $\mathbf{F}^{-1} \circ \mathbf{F} = \mathrm{Id}_{\mathbb{R}^3}$, we obtain

$$d(\mathbf{F}^{-1})_{\mathbf{F}(\mathbf{p})} \circ d\mathbf{F}_{\mathbf{p}} = d(\mathrm{Id}_{\mathbb{R}^3})_{\mathbf{p}} = \mathrm{Id}_{\mathbb{R}^3}.$$

Since $\mathrm{Id}_{\mathbb{R}^3}$ is one-to-one, $d\mathbf{F_p}$ is also one-to-one. Therefore $d(\mathbf{F} \circ \mathbf{x})_\mathbf{q} = d\mathbf{F_{x(q)}} \circ d\mathbf{x_q}$ is one-to-one for each $\mathbf{q}$, being composition of two injective maps.

Now since S and $\overline{S}$ have contact of order ≥ 1 at $\mathbf{p}$, there exist parametrizations $\mathbf{x}(u,v)$, $\overline{\mathbf{x}}(u,v)$ of S and $\overline{S}$, respectively, such that $\mathbf{x}_u = \overline{\mathbf{x}}_u$, $\mathbf{x}_v = \overline{\mathbf{x}}_v$. We showed that $(\mathbf{F} \circ \mathbf{x})(u,v)$, $(\mathbf{F} \circ \overline{\mathbf{x}})(u,v)$ is parametrizations for $\mathbf{F}(S)$ and $\mathbf{F}(\overline{S})$ at $\mathbf{F}(\mathbf{p})$. Furthermore,

$$(\mathbf{F} \circ \mathbf{x})_u = d\mathbf{F}(\mathbf{x}_u) = d\mathbf{F}(\overline{\mathbf{x}}_u) = (\mathbf{F} \circ \overline{\mathbf{x}})_u, \quad (\mathbf{F} \circ \mathbf{x})_v = d\mathbf{F}(\mathbf{x}_v) = d\mathbf{F}(\overline{\mathbf{x}}_v) = (\mathbf{F} \circ \overline{\mathbf{x}})_v.$$

Hence $\mathbf{F}(S)$ and $\mathbf{F}(\overline{S})$ are regular surfaces which have contact of order ≥ 1 at $\mathbf{F}(\mathbf{p})$.

e. Let $\mathbf{x}(u,v)$ and $\overline{\mathbf{x}}(u,v)$ be the parametrizations at $\mathbf{p}$ for S and $\overline{S}$, respectively, with $\mathbf{x}_u = \overline{\mathbf{x}}_u$, $\mathbf{x}_v = \overline{\mathbf{x}}_v$ and $\mathbf{x}(0,0) = \overline{\mathbf{x}}(0,0) = \mathbf{p}$. Since $\mathbf{x}$ and $\overline{\mathbf{x}}$ are differentiable, by Taylor's formula,

$$\mathbf{x}(u,v) = \mathbf{x}(0,0) + \mathbf{x}_u u + \mathbf{x}_v v + \mathbf{R}, \quad \overline{\mathbf{x}}(u,v) = \overline{\mathbf{x}}(0,0) + \overline{\mathbf{x}}_u u + \overline{\mathbf{x}}_v v + \overline{\mathbf{R}},$$

where the remainder $\mathbf{R}$, $\overline{\mathbf{R}}$ satisfies the condition

$$\lim_{r \to 0} \frac{\mathbf{R}}{r} = \lim_{r \to 0} \frac{\overline{\mathbf{R}}}{r} = \mathbf{0}, \quad r = \sqrt{u^2 + v^2}.$$

Let $\mathbf{N}$ be a common normal of the surfaces at $\mathbf{p}$, then

$$d = \langle \mathbf{x}(u,v) - \overline{\mathbf{x}}(u,v), \mathbf{N} \rangle = \langle \mathbf{R} - \overline{\mathbf{R}}, \mathbf{N} \rangle.$$

Therefore we have $d/r = \langle \mathbf{R}/r - \overline{\mathbf{R}}/r, \mathbf{N} \rangle \to \langle \mathbf{0}, \mathbf{N} \rangle = 0$ as $r \to 0$.

28. a. Given a differentiable map $f : S \to \mathbb{R}$ defined in regular surface S, we say that $\mathbf{p} \in S$ is a critical point of f if the differential $df_\mathbf{p} : T_\mathbf{p}(S) \to \mathbb{R}$ is not a surjective mapping. The image $f(\mathbf{p}) \in \mathbb{R}$ of a critical point is called a critical value of f. A point of $\mathbb{R}$ which is not a critical value is called a regular value of f.

b. Let a be a regular value of a differentiable function f on a regular surface S. We shall show that $f^{-1}(a) \subset S$ is a regular curve on S. For arbitrary point $\mathbf{p} \in f^{-1}(a)$, let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ be the parametrization for S at $\mathbf{p}$. Since a is a regular value, $df_\mathbf{p}$ is surjective. If $(f \circ \mathbf{x})_u = df_\mathbf{p}(\mathbf{x}_u)$ and $(f \circ \mathbf{x})_v = df_\mathbf{p}(\mathbf{x}_v)$ are both zero, then $df_\mathbf{p}(\mathbf{w}) = 0$ for every $\mathbf{w} \in T_\mathbf{p}(S)$ since $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent, which contradicts with the surjectivity of $f_\mathbf{p}$. Therefore one of the derivatives $(f \circ \mathbf{x})_u$ and $(f \circ \mathbf{x})_v$ are nonzero. We assume that $(f \circ \mathbf{x})_v \neq 0$.

Define a mapping $\mathbf{F} : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ as $\mathbf{F}(u,v) = (u, (f \circ \mathbf{x})(u,v))$, and we indicate by (w,t) the coordinates of a point in $\mathbb{R}^2$ where $\mathbf{F}$ takes its values. Since

$$d\mathbf{F_q} = \begin{pmatrix} 1 & 0 \\ (f \circ \mathbf{x})_u & (f \circ \mathbf{x})_v \end{pmatrix}, \quad \det(d\mathbf{F_q}) = (f \circ \mathbf{x})_v \neq 0, \quad \text{where } \mathbf{q} = \mathbf{x}^{-1}(\mathbf{p}),$$

We can apply the inverse function theorem, which guarantees the existence of neighborhoods V of $\mathbf{q}$ and W of $\mathbf{F}(\mathbf{q})$ such that $\mathbf{F} : V \to W$ is a diffeomorphism. It follows that the coordinate functions of $\mathbf{F}^{-1}$, i.e., the functions

$$u = w, \quad v = g(w,t), \quad (w,t) \in W$$

are differentiable. In particular, $v = g(w,a) = h(u)$ is a differentiable function defined in the projection of V onto the u axis. Now we assert that $\alpha : I = \pi_1(V) \to S$, $\alpha(u) = \mathbf{x}(u, h(u))$ is the parametrization for $f^{-1}(a)$ at $\mathbf{p}$. (i) α is differentiable since $\mathbf{x}$ and h are differentiable. (ii) Continuity of α is immediate. To see the injectivity, suppose that $\alpha(u_1) = \alpha(u_2)$ for some $u_1, u_2 \in I$. Since $\mathbf{x}$ is homeomorphic, $\mathbf{x}(u_1, h(u_1)) = \mathbf{x}(u_2, h(u_2))$ implies that $(u_1, h(u_1)) = (u_2, h(u_2))$, which leads $u_1 = u_2$. Hence α is injective. Finally, since $\alpha^{-1} = \pi_1 \circ \mathbf{x}^{-1}$ is continuous, α is a homeomorphism. (iii) Note that

$$d\alpha_u = \mathbf{x}_u(u, h(u)) + \mathbf{x}_v(u, h(u))h'(u) \neq \mathbf{0}$$

because $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent. Therefore $d\alpha_u$ is one-to-one for every $u \in \pi_1(V)$. We conclude that α is the parametrization for $f^{-1}(a)$ at $\mathbf{p}$ and that $f^{-1}(a)$ is a regular curve on S.

2-5. The First Fundamental Form; Area

1. First of all, note that if $\mathbf{w}$ is a tangent vector to the surface at the point $\mathbf{x}(u,v)$, given in the basis associated to $\mathbf{x}(u,v)$ by $\mathbf{w} = a\mathbf{x}_u + b\mathbf{x}_v$, then the first fundamental form is given by

$$\mathrm{I}(\mathbf{w}) = Ea^2 + 2Fab + Gb^2$$

where $E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle$, $F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle$, $G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle$ are the coefficients of the first fundamental form in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$.

a. Since $\mathbf{x}_u = (a\cos u \cos v, b\cos u \sin v, -c\sin u)$ and $\mathbf{x}_v = (-a\sin u \sin v, b\sin u \cos v, 0)$,

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = a^2 \cos^2 u \cos^2 v + b^2 \cos^2 u \sin^2 v + c^2 \sin^2 u,$$

$$F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = (-a^2 + b^2) \cos u \sin u \cos v \sin v,$$

$$G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = (a^2 \sin^2 v + b^2 \cos^2 v) \sin^2 u.$$

b. Since $\mathbf{x}_u = (a\cos v, b\sin v, 2u)$ and $\mathbf{x}_v = (-au\sin v, bu\cos v, 0)$,

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = a^2 \cos^2 v + b^2 \sin^2 v + 4u^2,$$

$$F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = (-a^2 + b^2) u \cos v \sin v,$$

$$G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = (a^2 \sin^2 v + b^2 \cos^2 v) u^2.$$

c. Since $\mathbf{x}_u = (a\cosh v, b\sinh v, 2u)$ and $\mathbf{x}_v = (au\sinh v, bu\cosh v, 0)$,

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = a^2 \cosh^2 v + b^2 \sinh^2 v + 4u^2,$$

$$F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = (a^2 + b^2) u \cosh v \sinh v,$$

$$G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = (a^2 \sinh^2 v + b^2 \cosh^2 v) u^2.$$

d. Since $\mathbf{x}_u = (a\cosh u \cos v, b\cosh u \sin v, c\sinh u)$ and $\mathbf{x}_v = (-a\sinh u \sin v, b\sinh u \cos v, 0)$,

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = a^2 \cosh^2 u \cos^2 v + b^2 \cosh^2 u \sin^2 v + c^2 \sinh^2 u,$$

$$F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = (-a^2 + b^2) \cosh u \sinh u \cos v \sin v,$$

$$G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = (a^2 \sin^2 v + b^2 \cos^2 v) \sinh^2 u.$$

2. The curve $P \cap \mathbb{S}^2$ is contained in the plane perpendicular to $(\sin\alpha, 0, -\cos\alpha)$, and the semimeridian $\varphi = \varphi_0$ is contained in the plane perpendicular to $(\sin\varphi_0, -\cos\varphi_0, 0)$. Since the angle between two curves is equal to the angle made by two planes, we have

$$\cos\beta = |\langle (\sin\alpha, 0, -\cos\alpha), (\sin\varphi_0, -\cos\varphi_0, 0) \rangle| = |\sin\alpha \sin\varphi_0| = \sin\alpha |\sin\varphi_0|.$$

We wrote $|\sin\alpha| = \sin\alpha$ because $0 < \alpha < \pi$.

3. Note that the parametrization for the stereographic projection is given by

$$\mathbf{x}(u,v) = \left(\frac{4u}{u^2 + v^2 + 4}, \frac{4v}{u^2 + v^2 + 4}, \frac{2(u^2 + v^2)}{u^2 + v^2 + 4} \right).$$

Since

$$\mathbf{x}_u = \frac{4}{(u^2 + v^2 + 4)^2}(-u^2 + v^2 + 4, -2uv, 4u), \quad \mathbf{x}_v = \frac{4}{(u^2 + v^2 + 4)^2}(-2uv, u^2 - v^2 + 4, 4v),$$

We have

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 16/(u^2 + v^2 + 4)^2,$$

$$F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0,$$

$$G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = 16/(u^2 + v^2 + 4)^2.$$

Therefore the first fundamental form for a tangent vector $\mathbf{w} = a\mathbf{x}_u + b\mathbf{x}_v$ is given by

$$\mathrm{I}(\mathbf{w}) = \frac{16(a^2 + b^2)}{(u^2 + v^2 + 4)^2}.$$

4. Since $\mathbf{x}_u = (\cos v, \sin v, 1)$ and $\mathbf{x}_v = (-u \sin v, u \cos v, -\tan v)$, the coefficients of the first fundamental form are

$$E = 2, \quad F = -\tan v, \quad G = u^2 + \tan^2 v.$$

Now consider a u-parameter curve $\alpha : I \to S$, $\alpha(t) = \mathbf{x}(u(t), v(t))$ where $u(t_1) = u_1 < u(t_2) = u_2$ and $v(t) = v_0$. Since $v'(t) \equiv 0$, the arc length s of α from $u = u_1$ to $u = u_2$ is given by

$$s = \int_{t_1}^{t_2} \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}\, dt = \int_{t_1}^{t_2} \sqrt{2}\, u'\, dt = \sqrt{2}(u_2 - u_1).$$

Therefore, since the length of the segment is independent of v_0, the two curves $\mathbf{x}(u_1, v)$ and $\mathbf{x}(u_2, v)$ determine segments of equal length on all curves $\mathbf{x}(u, \text{const.})$.

Note There is a typo in the problem. The two curves should be denoted as $\mathbf{x}(u_1, v)$ and $\mathbf{x}(u_2, v)$ instead of $\mathbf{x}(u, v_1)$ and $\mathbf{x}(u, v_2)$. This change is necessary because we are measuring segments along the u-parameter curves $(\mathbf{x}(u, \text{const.}))$, which must be bounded by two distinct v-parameter curves $(u = u_1$ and $u = u_2)$.

5. We can see that the map $\mathbf{x} : U \subset \mathbb{R}^2 \to S$, $\mathbf{x}(x, y) = (x, y, f(x, y))$ is the parametrization for the surface S given by $z = f(x, y)$. Then the area A of a bounded region R is

$$A = \iint_Q |\mathbf{x}_x \wedge \mathbf{x}_y|\, dx\, dy,$$

where $Q = \mathbf{x}^{-1}(R)$ is the normal projection of R onto the xy plane. Since $\mathbf{x}_x = (1, 0, f_x)$ and $\mathbf{x}_y = (0, 1, f_y)$, we have $|\mathbf{x}_x \wedge \mathbf{x}_y| = |(-f_x, -f_y, 1)| = \sqrt{1 + f_x^2 + f_y^2}$. Hence

$$A = \iint_Q \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy.$$

6. We first show that $\mathbf{x}(u, v) = (u \sin \alpha \cos v, u \sin \alpha \sin v, u \cos \alpha)$ is a parametrization of the cone with 2α as the angle of the vertex. (i) It is trivial that $\mathbf{x}$ is differentiable. (ii) Let $\mathbf{x}(u, v) = (u \sin \alpha \cos v, u \sin \alpha \sin v, u \cos \alpha) = (x, y, z)$. Since $u = \sqrt{x^2 + y^2 + z^2}$, u is a continuous function of (x, y, z). Furthermore, since $\sqrt{x^2 + y^2} = u \sin \alpha > 0$, we obtain

$$\cos v = \frac{x}{\sqrt{x^2 + y^2}}, \quad \sin v = \frac{y}{\sqrt{x^2 + y^2}}.$$

Therefore v is also a continuous function of (x, y, z) and $\mathbf{x}$ is a homeomorphism. (iii) Since

$$dx_{\mathbf{q}} = \begin{pmatrix} \sin \alpha \cos v & -u \sin \alpha \sin v \\ \sin \alpha \sin v & u \sin \alpha \cos v \\ \cos \alpha & 0 \end{pmatrix}, \quad \frac{\partial(x, y)}{\partial(u, v)} = u \sin^2 \alpha \neq 0,$$

the differential $dx_{\mathbf{q}}$ is one-to-one for each $\mathbf{q} \in U$. Therefore $\mathbf{x}$ is a parametrization.
Observe that $\mathbf{x}_u = (\sin \alpha \cos v, \sin \alpha \sin v, \cos \alpha)$ and $\mathbf{x}_v = (-u \sin \alpha \sin v, u \sin \alpha \cos v, 0)$, thus

$$E = 1, \quad F = 0, \quad G = u^2 \sin^2 \alpha.$$

Consider the curve $\alpha(t) = \mathbf{x}(u(t), v(t)) = \mathbf{x}(c \exp(v(t) \sin \alpha \cot \beta), v(t))$, whose tangent vector is $\alpha'(t) = \mathbf{x}_u u'(t) + \mathbf{x}_v v'(t)$. Note that

$$u'(t) = cv'(t) \sin \alpha \cot \beta \exp(v(t) \sin \alpha \cot \beta) = u(t)v'(t) \sin \alpha \cot \beta.$$

Let $\theta(t)$ denote the angle between the curve α and generators of the cone $(v = v_0)$ at the intersection $\alpha(t)$. Then

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$$\cos\theta(t) = \frac{\langle \mathbf{x}_u, \alpha' \rangle}{|\mathbf{x}_u||\alpha'|} = \frac{Eu' + Fv'}{\sqrt{E(u')^2 + 2Fu'v' + G(v')^2}}$$

$$= \frac{Euv' \sin\alpha \cot\beta}{\sqrt{E(uv' \sin\alpha \cot\beta)^2 + G(v')^2}} = \frac{Eu \sin\alpha \cot\beta}{\sqrt{Eu^2 \sin^2\alpha \cot^2\beta + G}}$$

$$= \frac{u \sin\alpha \cot\beta}{\sqrt{u^2 \sin^2\alpha \cot^2\beta + u^2 \sin^2\alpha}} = \frac{\cot\beta}{\sqrt{\cot^2\beta + 1}} = \operatorname{sgn}(\sin\beta)\cos\beta.$$

Therefore the curve $\mathbf{x}(u(t), v(t))$ intersects the generators of the cone at a constant angle $\theta(t) = \theta$.

 Note We can see that θ is not always equal to β. Specifically, $\theta = \beta$ for $\beta \in (0, \pi)$, and $\theta = \beta - \pi$ for $\beta \in (\pi, 2\pi)$.

7. Consider a quadrilateral formed by two u-parameter curves $v = v_1$, $v = v_2$ and two v-parameter curves $u = u_1$, $u = u_2$. The lengths of the opposite sides corresponding to the u-parameter curves are given by

$$s_1 = \int_{t_1}^{t_2} \sqrt{E(u(t), v_1)(u'(t))^2}\, dt = \int_{u_1}^{u_2} \sqrt{E(u, v_1)}\, du,$$

$$s_2 = \int_{t_1}^{t_2} \sqrt{E(u(t), v_2)(u'(t))^2}\, dt = \int_{u_1}^{u_2} \sqrt{E(u, v_2)}\, du$$

because $v' \equiv 0$ along the u-parameter curves. These values are equal for arbitrary u_1, u_2, v_1, v_2 if and only if

$$\int_{u_1}^{u_2} \left(\sqrt{E(u, v_1)} - \sqrt{E(u, v_2)} \right) du = 0 \quad \text{for arbitrary } u_1, u_2, v_1, v_2.$$

This is equivalent to the statement $\sqrt{E(u, v_1)} - \sqrt{E(u, v_2)} = 0$ for arbitrary u, v_1, v_2, which is equivalent to $E(u, v) = E(u)$. Similarly, we can show that lengths of the opposite sides corresponding to the v-parameter curves are always equal if and only if $G(u, v) = G(v)$. Therefore the coordinate curves of a parametrization $\mathbf{x}(u, v)$ constitute a Tchebyshef net if and only if $\partial E/\partial v = \partial G/\partial u = 0$.

8. Let $\mathbf{x}(u, v)$ be the parametrization constituting a Tchebyshef net. By Exercise 7, the coefficients of the first fundamental form satisfy the condition $E = E(u)$ and $G = G(v)$. Note that $E(u) = |x_u|^2 \neq 0$ and $G(v) = |x_v|^2 \neq 0$, because $d\mathbf{x_q} = (\mathbf{x}_u | \mathbf{x}_v)$ is one-to-one. We define

$$\overline{u}(u) = \int_{u_0}^{u} \sqrt{E(u)}\, du, \quad \overline{v}(v) = \int_{v_0}^{v} \sqrt{G(v)}\, dv.$$

Since $d\overline{u}/du = \sqrt{E(u)} \neq 0$ and $d\overline{v}/dv = \sqrt{G(v)} \neq 0$, there exists an inverse function $u(\overline{u})$ of $\overline{u}(u)$ and $v(\overline{v})$ of $\overline{v}(v)$, respectively. Now we reparametrize the coordinate neighborhood $\mathbf{x}$ to $\overline{\mathbf{x}}$ by

$$\overline{\mathbf{x}}(\overline{u}, \overline{v}) = \mathbf{x}(u(\overline{u}), v(\overline{v})).$$

Then

$$|\overline{\mathbf{x}}_{\overline{u}}| = \left| \mathbf{x}_u \frac{du}{d\overline{u}} \right| = |\mathbf{x}_u| \left| \frac{d\overline{u}}{du} \right|^{-1} = \sqrt{E(u)} \cdot \frac{1}{\sqrt{E(u)}} = 1 \implies \overline{E} = \langle \overline{\mathbf{x}}_{\overline{u}}, \overline{\mathbf{x}}_{\overline{u}} \rangle = 1.$$

Similarly, we obtain $\overline{G} = \langle \overline{\mathbf{x}}_{\overline{v}}, \overline{\mathbf{x}}_{\overline{v}} \rangle = 1$. Furthermore,

$$\overline{F} = \langle \overline{\mathbf{x}}_{\overline{u}}, \overline{\mathbf{x}}_{\overline{v}} \rangle = |\overline{\mathbf{x}}_{\overline{u}}||\overline{\mathbf{x}}_{\overline{v}}| \cos\theta = \cos\theta$$

where θ is the angle of the coordinate curves.

9. Let S be the surface of revolution obtained by rotating a regular connected plane curve C. We shall take xz plane as the plane of the curve and the z axis as the rotation axis. Let

$$x = f(v), \quad z = g(v), \quad a < v < b, \quad f(v) > 0$$

be a parametrization for C. We can suppose that the curve C is parametrized by an arc length, that is, $f'(v)^2 + g'(v)^2 = 1$. Now we obtain a parametrization for S by $\mathbf{x}: U \to S$,

$$\mathbf{x}(u, v) = (f(v)\cos u, f(v)\sin u, g(v))),$$

from the open set $U = \{(u, v) \in \mathbb{R}^2 \mid 0 < u < 2\pi, \; a < v < b\}$ into S. Since
$$\mathbf{x}_u = (-f(v) \sin u, f(v) \cos u, 0), \quad \mathbf{x}_v = (f'(v) \cos u, f'(v) \sin u, g'(v))),$$
The coefficients of the first fundamental form are
$$E = f(v)^2 = E(v), \quad F = 0, \quad G = f'(v)^2 + g'(v)^2 = 1.$$

10. Since $\mathbf{x}_\rho = (\cos \theta, \sin \theta)$ and $\mathbf{x}_\theta = (-\rho \sin \theta, \rho \cos \theta)$,
$$E = \langle \mathbf{x}_\rho, \mathbf{x}_\rho \rangle = 1, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0, \quad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = \rho^2.$$

11. a. We take the z axis as the rotation axis and the xz plane as the plane of the curve C. Let
$$x = \rho(s), \quad z = h(s), \quad 0 < s < l, \; \rho(s) > 0$$
be the parametrization for C, where s is the arc length, so $(\rho'(s))^2 + (h'(s))^2 = 1$. Then the parametrization for the surface of revolution S is given by
$$\mathbf{x}(\theta, s) = (\rho(s) \cos \theta, \rho(s) \sin \theta, h(s)), \quad 0 < s < l, \; 0 < \theta < 2\pi.$$

Since
$$\mathbf{x}_\theta = (-\rho(s) \sin \theta, \rho(s) \cos \theta, 0), \quad \mathbf{x}_s = (\rho'(s) \cos \theta, \rho'(s) \sin \theta, h'(s)),$$
we obtain
$$E = \rho(s)^2, \quad F = 0, \quad G = (\rho'(s))^2 + (h'(s))^2 = 1.$$

Therefore the area of S is
$$A = \int_0^l \int_0^{2\pi} \sqrt{EG - F^2} \, d\theta \, ds = \int_0^l \int_0^{2\pi} \rho(s) \, d\theta \, ds = 2\pi \int_0^l \rho(s) \, ds.$$

b. Consider the torus generated by rotating a circle of radius r about a straight line belonging to the plane of the circle and at a distance $a > r$ away from the center of the circle. Then the parametrization for the circle of radius r is given by
$$x = \rho(s) = a + r \cos \frac{s}{r}, \quad z = h(s) = r \sin \frac{s}{r}, \quad 0 < s < 2\pi r.$$

Note that s is the arc length because $(\rho'(s))^2 + (h'(s))^2 = 1$. By (a), the area of a torus is
$$A = 2\pi \int_0^{2\pi r} \rho(s) \, ds = 2\pi \int_0^{2\pi r} \left(a + r \cos \frac{s}{r} \right) ds = 4\pi^2 ra.$$
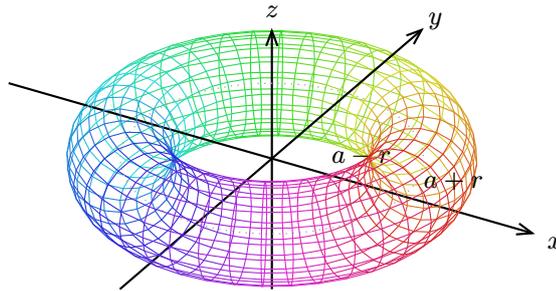
12. **Note** We suppose the natural situation, that is, $r > 0$ and $1/k(s) > r$.

The parametrization for the regular tube of radius r around a curve α is given by
$$\mathbf{x}(s, v) = \alpha(s) + r(\mathbf{n}(s) \cos v + \mathbf{b}(s) \sin v), \quad r = \text{const.} \neq 0, \; s \in I.$$

In the proof of Exercise 10 of Sec. 2-4, we obtained
$$\mathbf{x}_s \wedge \mathbf{x}_v = -r(1 - rk(s) \cos v)(\mathbf{b}(s) \sin v + \mathbf{n}(s) \cos v).$$

Since $|\mathbf{x}_s \wedge \mathbf{x}_v| = r(1 - rk(s) \cos v)$, the area of a regular tube is

$$A = \int_0^l \int_0^{2\pi} |\mathbf{x}_s \wedge \mathbf{x}_v| \, dv \, ds = \int_0^l \int_0^{2\pi} r(1 - rk(s)\cos v) \, dv \, ds = \int_0^l 2\pi r \, ds = 2\pi rl,$$

where l is the length of α.

13. **a.** We show that $\mathbf{x} : U \to S$, $\mathbf{x}(s, u) = (f(s)\cos u, f(s)\sin u, g(s) + cu)$ is a parametrization for S. (i) It is easy to see that $\mathbf{x}$ is differentiable. (ii) Let

$$\mathbf{x}(s, u) = (f(s)\cos u, f(s)\sin u, g(s) + cu) = (x, y, z).$$

From the equation $(f(s)\cos u, f(s)\sin u) = (x, y)$, we can determine $f(s)$ and u as

$$f(s) = \sqrt{x^2 + y^2}, \quad u = \arg\left(\frac{x + iy}{\sqrt{x^2 + y^2}}\right),$$

where $\arg : \{z \in \mathbb{C} \mid |z| = 1\} \to (0, 2\pi)$ is an inverse function of $z \mapsto e^{iz}$. So u is a continuous function of (x, y, z). Furthermore, $g(s) + cu = z$ yields

$$g(s) = z - cu = z - c \arg\left(\frac{x + iy}{\sqrt{x^2 + y^2}}\right).$$

Since $\alpha(s) = (f(s), g(s))$ is a parametrization for the regular plane curve C, α is a homeomorphism. Hence

$$s = \alpha^{-1}\left(\sqrt{x^2 + y^2}, z - c \arg\left(\frac{x + iy}{\sqrt{x^2 + y^2}}\right)\right)$$

is a continuous function of (x, y, z). Therefore $\mathbf{x}$ is a homeomorphism. (iii) Since

$$d\mathbf{x_q} = \begin{pmatrix} f'(s)\cos u & -f(s)\sin u \\ f'(s)\sin u & f(s)\cos u \\ g'(s) & c \end{pmatrix},$$

we have

$$\left(\frac{\partial(x, y)}{\partial(s, u)}\right)^2 + \left(\frac{\partial(x, z)}{\partial(s, u)}\right)^2 + \left(\frac{\partial(y, z)}{\partial(s, u)}\right)^2$$

$$= (f(s)f'(s))^2 + (cf'(s)\cos u + f(s)g'(s)\sin u)^2 + (cf'(s)\sin u - f(s)g'(s)\cos u)^2$$

$$= f(s)^2(f'(s)^2 + g'(s)^2) + c^2 f'(s)^2 = f(s)^2 + c^2 f'(s)^2 \geq f(s)^2 > 0,$$

which implies that $d\mathbf{x_q}$ is one-to-one for every $\mathbf{q} \in U$. Therefore $\mathbf{x}$ is a parametrization for S and we conclude that S is a regular surface.

b. The coordinate lines of the above parametrization are orthogonal if and only if

$$F = \langle \mathbf{x}_s, \mathbf{x}_u \rangle = (f'(s)\cos u, f'(s)\sin u, g'(s)) \cdot (-f(s)\sin u, f(s)\cos u, c) = cg'(s) = 0,$$

which occurs if and only if $c = 0$ ($\mathbf{x}(U)$ is a surface of revolution), or $g'(s) = 0$ for all $s \in (a, b)$ ($\mathbf{x}(U)$ is a piece of the standard helicoid).

14. **a.** Since $\operatorname{grad} f(\mathbf{p}) \in T_\mathbf{p}(S)$, it can be expressed as

$$\operatorname{grad} f(\mathbf{p}) = A\mathbf{x}_u(\mathbf{q}) + B\mathbf{x}_v(\mathbf{q})$$

for appropriate real numbers A and B where $\mathbf{q} = \mathbf{x}^{-1}(\mathbf{p})$. Recall that

$$\langle \operatorname{grad} f(\mathbf{p}), \mathbf{v} \rangle_\mathbf{p} = df_\mathbf{p}(\mathbf{v}) \quad \text{for all } \mathbf{v} \in T_\mathbf{p}(S).$$

Letting $\mathbf{v} = \mathbf{x}_u(\mathbf{q})$ in the equation above yields

$$\langle \operatorname{grad} f(\mathbf{p}), \mathbf{x}_u(\mathbf{q}) \rangle_\mathbf{p} = df_\mathbf{p}(\mathbf{x}_u(\mathbf{q})).$$

Since

$$\langle \operatorname{grad} f(\mathbf{p}), \mathbf{x}_u(\mathbf{q}) \rangle_{\mathbf{p}} = \langle A\mathbf{x}_u(\mathbf{q}) + B\mathbf{x}_v(\mathbf{q}), \mathbf{x}_u(\mathbf{q}) \rangle_{\mathbf{p}}$$

$$= A\langle \mathbf{x}_{u(\mathbf{q})}, \mathbf{x}_{u(\mathbf{q})} \rangle_{\mathbf{p}} + B\langle \mathbf{x}_v(\mathbf{q}), \mathbf{x}_u(\mathbf{q}) \rangle_{\mathbf{p}}$$

$$= AE + BF$$

and

$$f_u(\mathbf{p}) := (f \circ \mathbf{x})_u(\mathbf{q}) = d(f \circ \mathbf{x})_{\mathbf{q}}(\mathbf{e}_1) = df_{\mathbf{p}}(d\mathbf{x}_{\mathbf{q}}(\mathbf{e}_1)) = df_{\mathbf{p}}(\mathbf{x}_u(\mathbf{q})),$$

we obtain $f_u(\mathbf{p}) = AE + BF$. Similarly, letting $\mathbf{v} = \mathbf{x}_v(\mathbf{q})$ gives $f_v(\mathbf{p}) = AF + BG$. Hence

$$\begin{pmatrix} f_u \\ f_v \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \implies \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} f_u \\ f_v \end{pmatrix} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} f_u \\ f_v \end{pmatrix}$$

$$\implies A = \frac{f_u G - f_v F}{EG - F^2}, \quad B = \frac{f_v E - f_u F}{EG - F^2}$$

$$\implies \operatorname{grad} f = A\mathbf{x}_u + B\mathbf{x}_v = \frac{f_u G - f_v F}{EG - F^2}\mathbf{x}_u + \frac{f_v E - f_u F}{EG - F^2}\mathbf{x}_v.$$

b. Let θ be the angle between $\operatorname{grad} f(\mathbf{p})$ and $\mathbf{v} \in T_{\mathbf{p}}(S)$. Then

$$df_{\mathbf{p}}(\mathbf{v}) = \langle \operatorname{grad} f(\mathbf{p}), \mathbf{v} \rangle = |\operatorname{grad} f(\mathbf{p})||\mathbf{v}| \cos\theta = |\operatorname{grad} f(\mathbf{p})| \cos\theta$$

is maximum if and only if $\cos\theta = 1$, and this means $\theta = 0$ and $\mathbf{v} = (\operatorname{grad} f(\mathbf{p}))/|\operatorname{grad} f(\mathbf{p})|$.

c. The level curve C is given by an inverse image $f^{-1}(a)$ for a constant a. Since $\operatorname{grad} f(\mathbf{p}) \neq \mathbf{0}$ for every point $\mathbf{p} \in C$, $df_{\mathbf{p}}(\mathbf{v}) = \langle \operatorname{grad} f(\mathbf{p}), \mathbf{v} \rangle_{\mathbf{p}}$ is surjective for every point $\mathbf{p} \in C$. Hence there is no critical point, and a is a regular value of f. By Exercise 28(b) of Sec. 2-4, we conclude that $C = f^{-1}(a)$ is a regular curve on S.

15. a. Let C_1 be the regular curve on $\mathbf{x}(U) \subset S$ given by the equation $\varphi(u, v) = c_1$. In other words, for every point $\mathbf{p} \in C_1$, we have $\varphi(\mathbf{x}^{-1}(\mathbf{p})) = c_1$. Let $\overline{\alpha} : I \to U$ be the parametrization for $\mathbf{x}^{-1}(C_1) = \varphi^{-1}(c_1)$ at $\mathbf{x}^{-1}(\mathbf{p})$, then the mapping $\alpha : I \to \mathbf{x}(U)$ defined by $\alpha(t) = \mathbf{x}(\overline{\alpha}(t))$ becomes the parametrization for C_1 at $\mathbf{p}$. Since $\varphi(\overline{\alpha}(t)) = c_1$, differentiating with respect to t gives

$$(\varphi_u, \varphi_v) \cdot \overline{\alpha}'(t) = 0.$$

Hence we obtain $\overline{\alpha}'(t) = \lambda_1(t)(\varphi_v, -\varphi_u)$ for nonzero $\lambda_1(t)$. Then

$$\alpha'(t) = d\mathbf{x}_{\overline{\alpha}(t)}(\overline{\alpha}'(t)) = \lambda_1 d\mathbf{x}_{\mathbf{q}}(\varphi_v, -\varphi_u) = \lambda_1(\mathbf{x}_u \varphi_v - \mathbf{x}_v \varphi_u).$$

Similarly, let C_2 be the curve given by $\psi(u, v) = c_2$, and let $\beta : J \to \mathbf{x}(U)$ be the parametrization for C_2 at $\mathbf{p}$, then we have

$$\beta'(t) = \lambda_2(\mathbf{x}_u \psi_v - \mathbf{x}_v \psi_u).$$

Therefore two curves are orthogonal if and only if the inner product $\langle \alpha'(t), \beta'(t) \rangle$ is zero, that is,

$$\langle \alpha'(t), \beta'(t) \rangle = \lambda_1 \lambda_2 \langle \mathbf{x}_u \varphi_v - \mathbf{x}_v \varphi_u, \mathbf{x}_u \psi_v - \mathbf{x}_v \psi_u \rangle$$

$$= \lambda_1 \lambda_2 (\langle \mathbf{x}_u, \mathbf{x}_u \rangle \varphi_v \psi_v - \langle \mathbf{x}_u, \mathbf{x}_v \rangle \varphi_u \psi_v + \varphi_v \psi_u) + \langle \mathbf{x}_v, \mathbf{x}_v \rangle \varphi_u \psi_u)$$

$$= \lambda_1 \lambda_2 (E\varphi_v \psi_v - F(\varphi_u \psi_v + \varphi_v \psi_u) + G\varphi_u \psi_u)$$

$$= 0.$$

Since λ_1 and λ_2 are nonzero, it is equivalent to $E\varphi_v \psi_v - F(\varphi_u \psi_v + \varphi_v \psi_u) + G\varphi_u \psi_u = 0$.

Note In order to be orthogonal, it is sufficient to satisfy the equation above only for the intersections of two families. For example, if two curves on the plane $(E = 1, F = 0, G = 1)$ are given by

$$\varphi(u, v) = u^2 + v^2 = 1, \quad \psi(u, v) = u^2 + v^2 = 2,$$

Then

$$E\varphi_v \psi_v - F(\varphi_u \psi_v + \varphi_v \psi_u) + G\varphi_u \psi_u = 4v^2 - 4u^2 \neq 0.$$

However, these two curves never meet, so they are orthogonal.

b. The parametrization $\mathbf{x} : U \to S$ for the helicoid is given by $\mathbf{x}(u, v) = (v \cos u, v \sin u, au)$. Then

$$\mathbf{x}_u = (-v\sin u, v\cos), \quad \mathbf{x}_v = (\cos u, \sin u, 0)$$

implies that $E = v^2 + a^2, F = 0, G = 1$. Furthermore, let

$$\varphi(u,v) = v\cos u, \quad \psi(u,v) = (v^2 + a^2)\sin^2 u$$

then we have

$$E\varphi_v\psi_v - F(\varphi_u\psi_v + \varphi_v\psi_u) + G\varphi_u\psi_u$$
$$= (v^2 + a^2)\cdot\cos u\cdot 2v\sin^2 u + 1\cdot(-v\sin u)\cdot 2(v^2 + a^2)\sin u\cos u = 0.$$

By (a), two families of regular curves are orthogonal.

3-2. The Definition of the Gauss Map and Its Fundamental Properties

1. At a hyperbolic point $\mathbf{p}$ of regular surface S, we have $\det(d\mathbf{N_p}) = k_1 k_2 < 0$. Let $\mathbf{e}_1, \mathbf{e}_2$ be the eigenvectors of $d\mathbf{N_p}$. Then the second fundamental form at $\mathbf{p}$ is given by

$$\mathbf{II_p}(\mathbf{w}) = k_1\xi^2 + k_2\eta^2,$$

where $\mathbf{w} = \xi\mathbf{e}_1 + \eta\mathbf{e}_2 \in T_\mathbf{p}(S)$. The asymptotic directions satisfy $k_n(\mathbf{w}) = \mathbf{II_p}(\mathbf{w}) = 0$, which is represented by

$$\eta = \pm\sqrt{-\frac{k_1}{k_2}}\,\xi.$$

We can see that principal directions $\mathbf{e}_1, \mathbf{e}_2$ bisect the asymptotic directions above.

2. Suppose that the surface S is tangent to a plane P along a regular curve C. Given a point $\mathbf{p} \in C$, let $\alpha : (-\varepsilon, \varepsilon) \to C$ be a parametrization of C at $\mathbf{p}$. Observe that the unit normal vector

$$\mathbf{N}(t) = (\mathbf{N} \circ \alpha)(t)$$

is constant along the curve, since it is perpendicular to the plane P. Hence $N'(t) = \mathbf{0}$. By Prop. 3, C is a line of curvature of S, and the corresponding principal curvature is 0. Therefore k_1 or k_2 is zero, that is, $K = \det(d\mathbf{N_p}) = 0$. So we conclude that $\mathbf{p}$ is either parabolic or planar.

3. Let k_n be the normal curvature of C at $\mathbf{p}$. Then $k_n = k\cos\theta = k\langle\mathbf{n}, \mathbf{N}\rangle$, so $|k| \geq |k_n|$. Note that

$$k_n = k_1\cos^2\theta + k_2\sin^2\theta,$$

where θ is the angle from $\mathbf{e}_1$ to the tangent vector of C at $\mathbf{p}$. Since $K = k_1 k_2 > 0$, k_1 and k_2 have the same sign. Therefore k_n is the number between k_1 and k_2, and we conclude that

$$|k| \geq |k_n| \geq \min(|k_1|, |k_2|).$$

4. No. Consider the surface $S = \{(x,y,z) \in \mathbb{R}^3 \mid z = 0\}$ and regular curve

$$C = \{(x,y,z) \mid x^2 + y^2 = 1/4, \ z = 0\} \subset S.$$

Observe that S has the property that $|k_1| = |k_2| = 0 \leq 1$ everywhere, but the curvature $k = 2$ of a curve C does not satisfy $|k| \leq 1$.

5. Let $\mathbf{e}_1, \mathbf{e}_2$ be the eigenvectors of $d\mathbf{N_p}$, and φ be an angle between $\mathbf{e}_1$ and fixed direction $\mathbf{a}$. Then the normal curvature is given by

$$k_n(\theta) = k_1\cos^2(\theta + \varphi) + k_2\sin^2(\theta + \varphi).$$

Hence

$$\int_0^\pi k_n(\theta)\, d\theta = \int_0^\pi \left(k_1 \cos^2(\theta + \varphi) + k_2 \sin^2(\theta + \varphi) \right) d\theta$$

$$= \int_\varphi^{\pi + \varphi} \left(k_1 \cos^2 \psi + k_2 \sin^2 \psi \right) d\psi = \int_0^\pi \left(k_1 \cos^2 \psi + k_2 \sin^2 \psi \right) d\psi$$

$$= \frac{\pi}{2} k_1 + \frac{\pi}{2} k_2 = \pi \left(\frac{k_1 + k_2}{2} \right) = \pi H$$

and

$$H = \frac{1}{\pi} \int_0^\pi k_n(\theta)\, d\theta.$$

6. Let $\mathbf{e}_1, \mathbf{e}_2$ be the eigenvectors of $d\mathbf{N_p}$. We can suppose that $\theta, \theta + \pi/2$ is the angle between $\mathbf{e}_1$ and pair of orthogonal directions, respectively. Then the sum of the normal curvatures is

$$k_n(\theta) + k_n\left(\theta + \frac{\pi}{2}\right) = (k_1 \cos^2 \theta + k_2 \sin^2 \theta) + \left(k_1 \cos^2\left(\theta + \frac{\pi}{2}\right) + k_2 \sin^2\left(\theta + \frac{\pi}{2}\right) \right)$$

$$= k_1 \cos^2 \theta + k_2 \sin^2 \theta + k_1 \sin^2 \theta + k_2 \cos^2 \theta = k_1 + k_2 = 2H,$$

which is constant.

7. If the mean curvature $(k_1 + k_2)/2$ is zero at a nonplanar point, then we have $k_1 = -k_2 \neq 0$. Let $\mathbf{e}_1, \mathbf{e}_2$ be the eigenvectors of $d\mathbf{N_p}$ and $\mathbf{w} = \xi \mathbf{e}_1 + \eta \mathbf{e}_2 \in T_\mathbf{p}(S)$, then the second fundamental form at $\mathbf{p}$ is given by

$$\mathbf{II_p}(\mathbf{w}) = k_1 \xi^2 + k_2 \eta^2.$$

The asymptotic direction satisfies $k_n(\mathbf{w}) = \mathbf{II_p}(\mathbf{w}) = 0$, which is equivalent to

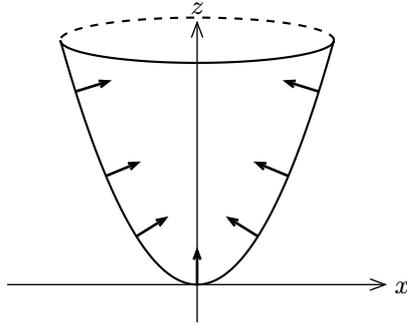$$\eta = \pm \sqrt{-\frac{k_1}{k_2}} \xi = \pm \xi.$$

Therefore the point $\mathbf{p}$ has two orthogonal asymptotic directions.

8. **a.** Let $\mathbf{x} : U = \mathbb{R}^2 \to S$, $\mathbf{x}(x, y) = (x, y, x^2 + y^2)$ be a parametrization for the paraboloid of revolution. Then $\mathbf{x}_u = (1, 0, 2x)$ and $\mathbf{x}_y = (1, 2y, 0)$, so $\mathbf{x}_u \wedge \mathbf{x}_y = (-2x, -2y, 1)$ and

$$\mathbf{N}(x, y, z) = \frac{(-2x, -2y, -1)}{\sqrt{4x^2 + 4y^2 + 1}}.$$

The image of the Gauss map is

$$\mathbf{N}(S) = \left\{ (x, y, z) \in \mathbb{S}^2 \mid z > 0 \right\}.$$

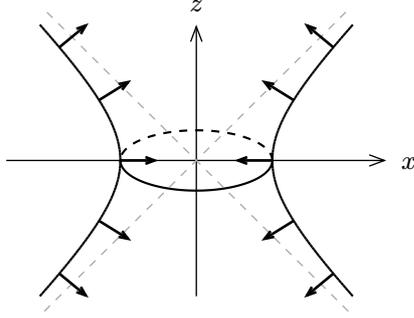**b.** If $\alpha(t) = (x(t), y(t), z(t))$ is a parametrized curve in hyperboloid of revolution S, then

$$xx' + yy' - zz' = 0,$$

which shows that the vector $(x, y, -z)$ is normal to S at the point (x, y, z). Therefore $\mathbf{N}(x, y, z) = (-x, -y, z)/\sqrt{x^2 + y^2 + z^2}$. In particular, the image of the Gauss map is

$$\mathbf{N}(S) = \left\{ (x, y, z) \in \mathbb{S}^2 \mid |z| > 1/\sqrt{2} \right\}.$$
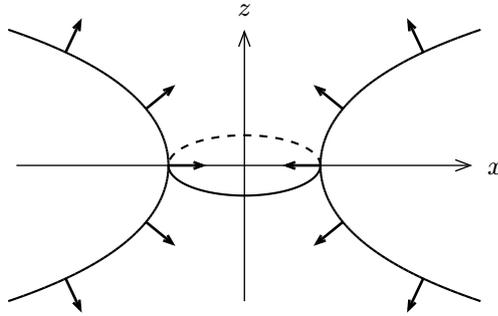
c. If $\alpha(t) = (x(t), y(t), z(t))$ is a parametrized curve in catenoid S, then

$$xx' + yy' - \cosh z \sinh z \, z' = 0,$$

which shows that the vector $(x, y, -\cosh z \sinh z)$ is normal to S at the point (x, y, z). Hence $\mathbf{N}(x, y, z) = (-x, -y, \cosh z \sinh z)/\sqrt{x^2 + y^2 + \cosh^2 z \sinh^2 z}$. The image of the Gauss map is

$$\mathbf{N}(S) = \{(x, y, z) \in \mathbb{S}^2 \mid |z| \neq 1\}.$$

9. **a.** Since the surface S contains no planar or parabolic points, $\det(d\mathbf{N_p}) \neq 0$ for every $\mathbf{p} \in S$. The parametrized regular curve $\alpha : I \to S$ satisfies $\alpha'(t) \neq \mathbf{0}$ for every $t \in I$. To show that $\mathbf{N} \circ \alpha : I \to \mathbb{S}^2$ is a parametrized regular curve, it must be shown that $(\mathbf{N} \circ \alpha)'(t) \neq \mathbf{0}$ for every $t \in I$. Since $(\mathbf{N} \circ \alpha)'(t) = d\mathbf{N}_{\alpha(t)}(\alpha'(t))$, where $\det(d\mathbf{N}_{\alpha(t)}) \neq 0$ and $\alpha'(t) \neq \mathbf{0}$, we obtain $(\mathbf{N} \circ \alpha)'(t) \neq \mathbf{0}$ for every $t \in I$.

b. Suppose that α is parametrized by arc length and let $\beta = \mathbf{N} \circ \alpha$. Since $C = \alpha(I)$ is a line of curvature, $\beta'(s) = (\mathbf{N} \circ \alpha)'(s) = \lambda(s)\alpha(s)$. Also, $\beta''(s) = \lambda'(s)\alpha(s) + \lambda(s)\alpha'(s)$, hence

$$k_N = \frac{|\beta' \wedge \beta''|}{|\beta'|^3} = \frac{|\lambda|^2 |\alpha \wedge \alpha'|}{|\lambda|^3 |\alpha|^3} = \frac{1}{|\lambda|} \frac{|\alpha \wedge \alpha'|}{|\alpha|^3} = \frac{1}{|\lambda|} k.$$

The normal curvature is given by $k_n = -\lambda(s)$. Therefore

$$k = |\lambda| k_N = |k_n| k_N = |k_n k_N|.$$

10. Choose arbitrary $\mathbf{p} \in C$, and let $\alpha : I \to C$ be an arc-length parametrization of the curve C at $\mathbf{p}$. The osculating plane of a curve C makes a constant angle with the tangent plane of S along C, so $\langle \mathbf{N}(s), \mathbf{b}(s) \rangle = \text{const.}$ for every $s \in I$. Since C is a line of curvature, $\mathbf{N}'(s) = \lambda(s)\alpha'(s)$. Hence

$$\langle \mathbf{N}', \mathbf{b} \rangle + \langle \mathbf{N}, \mathbf{b}' \rangle = 0 \implies \langle \lambda \alpha', \mathbf{b} \rangle + \langle \mathbf{N}, \tau \mathbf{n} \rangle = 0.$$

Since $\alpha' = \mathbf{t} \perp \mathbf{b}$ everywhere, $\langle \lambda \alpha', \mathbf{b} \rangle \equiv 0$, so we obtain $\tau \langle \mathbf{N}, \mathbf{n} \rangle \equiv 0$. However, C is nowhere tangent to an asymptotic direction, thus

$$k_n(\alpha') = \langle \mathbf{N}, k\mathbf{n} \rangle(\alpha') \neq 0.$$

As $\langle \mathbf{N}, \mathbf{n} \rangle \neq 0$ implies $\tau \equiv 0$ in the neighborhood I of $\mathbf{p}$, we conclude that C is a plane curve.

11. **Note** If p is an umbilical point of S, then every pair of directions r and r' make the minimum angle $\pi/2$, which is contradiction to conclusion that there exists an unique pair of directions in $T_{\mathbf{p}}(S)$. So we suppose that p is not an umbilical point.

Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the orthonormal basis of $T_\mathbf{p}(S)$ determined by $d\mathbf{N}_\mathbf{p}(\mathbf{e}_1) = -k_1\mathbf{e}_1$, $d\mathbf{N}_\mathbf{p}(\mathbf{e}_2) = -k_2\mathbf{e}_2$. Let θ and φ be the angles that a pair of directions r and r' make with $\mathbf{e}_1$. r and r' are conjugate if and only if the vectors

$$\mathbf{w}_1 = \mathbf{e}_1\cos\theta + \mathbf{e}_2\sin\theta, \quad \mathbf{w}_2 = \mathbf{e}_1\cos\varphi + \mathbf{e}_2\sin\varphi$$

are conjugate, that is,

$$0 = \langle d\mathbf{N}_\mathbf{p}(\mathbf{w}_1), \mathbf{w}_2\rangle = -k_1\cos\theta\cos\varphi - k_2\sin\theta\sin\varphi = 0.$$

Now restrict the range of θ to $\theta \in [0, \pi)$, then the corresponding φ for each θ is contained in $[\pi/2, 3\pi/2]$. We want to find the pair of angles (θ, φ) such that $\varphi - \theta$ is minimum.

If $\theta = 0$ or $\theta = \pi/2$, then we have $\varphi - \theta = \pi/2$.

If $\theta \in (0, \pi/2)$, then $\tan\theta > 0$ and $\tan\varphi < 0$. Let $\alpha = \tan\theta$, $\beta = -\tan\varphi$. Since $\alpha\beta = k_1/k_2$,

$$\tan(\varphi - \theta) = \frac{\tan\varphi - \tan\theta}{1 + \tan\varphi\tan\theta} = \frac{\alpha + \beta}{k_1/k_2 - 1}.$$

If $\theta \in (\pi/2, \pi)$, then $\tan\theta < 0$ and $\tan\varphi > 0$. Let $\alpha = -\tan\theta$, $\beta = \tan\varphi$. Since $\alpha\beta = k_1/k_2$,

$$\tan(\varphi - \theta) = \frac{\tan\varphi - \tan\theta}{1 + \tan\varphi\tan\theta} = \frac{\alpha + \beta}{1 - k_1/k_2}.$$

Now consider the cases $k_1 \geq k_2$ and $k_1 < k_2$.

If $k_1 > k_2$, then $k_1/k_2 > 1$; so $\tan(\varphi - \theta) > 0$ at $\theta \in (0, \pi/2)$ and $\tan(\varphi - \theta) > 0$ at $\theta \in (\pi/2, \pi)$. This means that

$$\varphi - \theta \in \begin{cases} (0, \pi/2) & \text{for } \theta \in (0, \pi/2) \\ \{\pi/2\} & \text{for } \theta = 0, \pi/2 \\ (\pi/2, \pi) & \text{for } \theta \in (\pi/2, \pi). \end{cases}$$

So we consider the case $\theta \in (0, \pi/2)$. By AM–GM inequality,

$$\tan(\varphi - \theta) = \frac{\alpha + \beta}{k_1/k_2 - 1} \geq \frac{2\sqrt{\alpha\beta}}{k_1/k_2 - 1} = \frac{2\sqrt{k_1/k_2}}{k_1/k_2 - 1}$$

when $\alpha = \beta$. $\tan\theta = -\tan\varphi$ implies $\theta + \varphi = \pi$, thus r and r' are symmetric with respect to the principal direction $\mathbf{e}_2$.

Otherwise, if $k_1 < k_2$, then $k_1/k_2 < 1$; so $\tan(\varphi - \theta) < 0$ at $\theta \in (0, \pi/2)$ and $\tan(\varphi - \theta) < 0$ at $\theta \in (\pi/2, \pi)$. This means that

$$\varphi - \theta \in \begin{cases} (\pi/2, \pi) & \text{for } \theta \in (0, \pi/2) \\ \{\pi/2\} & \text{for } \theta = 0, \pi/2 \\ (0, \pi/2) & \text{for } \theta \in (\pi/2, \pi). \end{cases}$$

So we consider the case $\theta \in (\pi/2, \pi)$ By AM–GM inequality,

$$\tan(\varphi - \theta) = \frac{\alpha + \beta}{1 - k_1/k_2} \geq \frac{2\sqrt{\alpha\beta}}{1 - k_1/k_2} = \frac{2\sqrt{k_1/k_2}}{1 - k_1/k_2}$$

when $\alpha = \beta$. $-\tan\theta = \tan\varphi$ implies $\theta + \varphi = 2\pi$, thus r and r' are symmetric with respect to the principal direction $\mathbf{e}_1$.

12. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the orthonormal basis of $T_\mathbf{p}(S)$ determined by $d\mathbf{N}_\mathbf{p}(\mathbf{e}_1) = -k_1\mathbf{e}_1$, $d\mathbf{N}_\mathbf{p}(\mathbf{e}_2) = -k_2\mathbf{e}_2$. Let θ and φ be the angles that a pair of directions r and r' make with $\mathbf{e}_1$. r and r' are conjugate if and only if the vectors

$$\mathbf{w}_1 = \mathbf{e}_1\cos\theta + \mathbf{e}_2\sin\theta, \quad \mathbf{w}_2 = \mathbf{e}_1\cos\varphi + \mathbf{e}_2\sin\varphi$$

are conjugate, that is,

$$0 = \langle d\mathbf{N}_\mathbf{p}(\mathbf{w}_1), \mathbf{w}_2\rangle = -k_1\cos\theta\cos\varphi - k_2\sin\theta\sin\varphi = 0.$$

We now describe the geometric construction to find the conjugate direction r' of r. Given straight line r through the origin, consider the intersection points $\mathbf{q}_1, \mathbf{q}_2$ of r with the Dupin indicatrix. We assert

that the tangent lines of the Dupin indicatrix at $\mathbf{q}_1$ and $\mathbf{q}_2$ are parallel, and their common direction r' is conjugate to r.
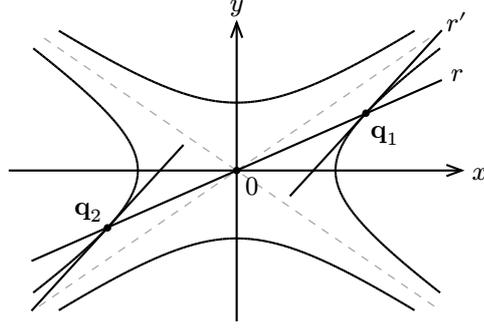
By considering the curve $(x(t), y(t))$ contained in the Dupin indicatrix, we have $k_1 x^2 + k_2 y^2 = \pm 1$. Differentiating gives $2k_1 x x' + 2k_2 y y' = 0$, so $\langle (k_1 x, k_2 y), (x', y') \rangle = 0$. This means that at the point (x, y), the normal vector is $(k_1 x, k_2 y)$. Hence the normal vector at $\mathbf{q}_1$ and $\mathbf{q}_2$ is $(k_1 \cos\theta, k_2 \sin\theta)$. Observe that $\langle (k_1 \cos\theta, k_2 \sin\theta), (\cos\varphi, \sin\varphi) \rangle = 0$, thus the tangent vectors at $\mathbf{q}_1$ and $\mathbf{q}_2$ are parallel to $(\cos\varphi, \sin\varphi)$, which is the direction of r'.

13. Given point $\mathbf{p}$ on the asymptotic curve C, let $\alpha : (-\varepsilon, \varepsilon) \to S$ be the arc-length parametrization with $\mathbf{p} = \alpha(0)$. We have $k_n = k\langle \mathbf{n}, \mathbf{N} \rangle = 0$ in the asymptotic curve. Since curvature k is nowhere zero, $\langle \mathbf{n}, \mathbf{N} \rangle \equiv 0$. Obviously $\langle \mathbf{t}, \mathbf{N} \rangle = 0$, therefore $\mathbf{N}$ and $\mathbf{b}$ are parallel and $\mathbf{N} = \pm \mathbf{b}$. Let $\mathbf{e}_1, \mathbf{e}_2$ be the eigenvectors of $d\mathbf{N_p}$, and let $\alpha'(0) = \mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta$, then

$$\mathbf{N}'(0) = d\mathbf{N_p}(\alpha'(0)) = -\mathbf{e}_1 k_1 \cos\theta - \mathbf{e}_2 k_2 \sin\theta, \quad k_n = k_1 \cos^2\theta + k_2 \sin^2\theta = 0.$$

Therefore

$$\tau^2 = |\mathbf{N}'(0)|^2 = k_1^2 \cos^2\theta + k_2^2 \sin^2\theta = k_1(k_1 \cos^2\theta) + k_2(k_2 \sin^2\theta)$$
$$= k_1(-k_2 \sin^2\theta) + k_2(-k_1 \cos^2\theta) = -k_1 k_2.$$

So absolute value of the torsion is given by

$$|\tau| = \sqrt{-k_1 k_2} = \sqrt{-K}.$$

14. Let $\mathbf{N}_1, \mathbf{N}_2$ be the normal vectors of S_1 and S_2 at $\mathbf{p}$. Since θ is the angle between $\mathbf{N}_1$ and $\mathbf{N}_2$,

$$|\lambda_1 \mathbf{N}_2 - \lambda_2 \mathbf{N}_1| = \sqrt{\lambda_1^2 + \lambda_2^2 - 2\lambda_1 \lambda_2 \cos\theta}.$$

Furthermore, $\lambda_1 = k\langle \mathbf{n}, \mathbf{N}_1 \rangle$ and $\lambda_2 = k\langle \mathbf{n}, \mathbf{N}_2 \rangle$ implies

$$|\lambda_1 \mathbf{N}_2 - \lambda_2 \mathbf{N}_1| = |k| |\langle \mathbf{n}, \mathbf{N}_1 \rangle \mathbf{N}_2 - \langle \mathbf{n}, \mathbf{N}_2 \rangle \mathbf{N}_1|$$
$$= |k| |\mathbf{n} \wedge (\mathbf{N}_1 \wedge \mathbf{N}_2)|$$
$$= |k| |\mathbf{N}_1 \wedge \mathbf{N}_2| = |k| |\sin\theta|.$$

Therefore

$$k^2 \sin^2\theta = |\lambda_1 \mathbf{N}_2 - \lambda_2 \mathbf{N}_1|^2 = \lambda_1^2 + \lambda_2^2 - 2\lambda_1 \lambda_2 \cos\theta.$$

15. Let $\mathbf{N}_1(\mathbf{p})$ and $\mathbf{N}_2(\mathbf{p})$ be the Gauss maps of S_1 and S_2 at $\mathbf{p}$, then $\cos\theta(\mathbf{p}) = \langle \mathbf{N}_1(\mathbf{p}), \mathbf{N}_2(\mathbf{p}) \rangle$.
Suppose that C is a line of curvature of S_2. Let $\alpha(t)$ be the parametrization of the regular curve C. Since C is also a line of curvature of S_1,

$$\frac{d}{dt} \cos\theta(\alpha(t)) = \frac{d}{dt} \langle \mathbf{N}_1(t), \mathbf{N}_2(t) \rangle = \langle \mathbf{N}_1'(t), \mathbf{N}_2(t) \rangle + \langle \mathbf{N}_1(t), \mathbf{N}_2'(t) \rangle$$
$$= \lambda_1(t) \langle \alpha'(t), \mathbf{N}_2(t) \rangle + \lambda_2(t) \langle \mathbf{N}_1(t), \alpha'(t) \rangle = 0.$$

Therefore $\theta(\mathbf{p})$ is constant in C.
Conversely, suppose that $\theta(\mathbf{p})$ is constant in C. Then $\cos\theta(\mathbf{p})$ is also constant, so

$$0 = \frac{d}{dt} \cos\theta(\alpha(t)) = \frac{d}{dt} \langle \mathbf{N}_1(t), \mathbf{N}_2(t) \rangle = \langle \mathbf{N}_1'(t), \mathbf{N}_2(t) \rangle + \langle \mathbf{N}_1(t), \mathbf{N}_2'(t) \rangle.$$

Since

$$\langle \mathbf{N}'_1(t), \mathbf{N}_2(t) \rangle = \lambda_1(t)\langle \alpha'(t), \mathbf{N}_2(t) \rangle = 0,$$

we obtain $\langle \mathbf{N}_1(t), \mathbf{N}'_2(t) \rangle = 0$. Note that $|\mathbf{N}_2(t)| = 1$ implies $\langle \mathbf{N}_2(t), \mathbf{N}'_2(t) \rangle = 1$, so $\mathbf{N}'_2(t)$ is perpendicular to both $\mathbf{N}_1(t)$ and $\mathbf{N}_2(t)$. If $\mathbf{N}_1(t)$ and $\mathbf{N}_2(t)$ are linearly independent, then $\mathbf{N}'_2(t)$ is parallel to $\mathbf{N}_1(t) \wedge \mathbf{N}_2(t)$, which is parallel to $\alpha'(t)$. Hence we can write

$$\mathbf{N}'_2(t) = \lambda_2(t)\alpha'(t).$$

Otherwise, if $\mathbf{N}_1(t)$ and $\mathbf{N}_2(t)$ are linearly dependent, then $\mathbf{N}_1(t) = \pm\mathbf{N}_2(t)$ implies $\mathbf{N}'_2(t) = \pm\mathbf{N}'_1(t) = \pm\lambda_1(t)\alpha'(t) = \lambda_2(t)\alpha'(t)$. In either case, we can obtain $\mathbf{N}'_2(t) = \lambda_2(t)\alpha'(t)$. Note that $\lambda_2(t)$ is differentiable because $\mathbf{N}'_2(t)$ and $\alpha'(t) \neq 0$ are differentiable. Thus C is a line of curvature of S_2.

16. Let S_1 be a torus, and S_2 be a plane containing its axis. Then S_1 and S_2 intersect along a regular curve C (which is a line of curvature of torus) and make a constant angle $\theta(\mathbf{p}) = \pi/2$. Note that C is a line of curvature of S_2 because $\mathbf{N}'_2(t) \equiv 0$ on plane. By Exercise 15, C is a line of curvature of S_1.
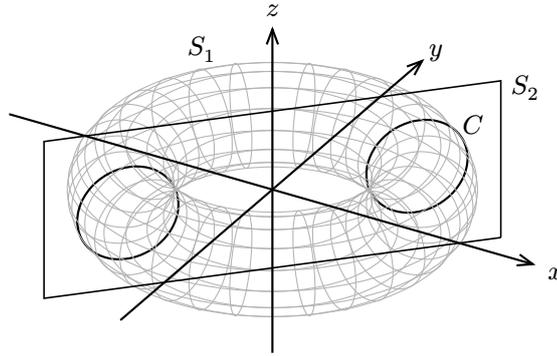
17. $H \equiv 0$ on S and S has no planar points, thus $k_1 + k_2 = 0$ and $d\mathbf{N}_\mathbf{p} \neq 0$ at every point of S. Let $\mathbf{e}_1, \mathbf{e}_2$ be the orthonormal basis of $T_\mathbf{p}(S)$ determined by $d\mathbf{N}_\mathbf{p}(\mathbf{e}_1) = -k_1\mathbf{e}_1$, $d\mathbf{N}_\mathbf{p}(\mathbf{e}_2) = -k_2\mathbf{e}_2$. Let

$$\mathbf{w}_1 = a_1\mathbf{e}_1 + a_2\mathbf{e}_2, \quad \mathbf{w}_2 = b_1\mathbf{e}_1 + b_2\mathbf{e}_2,$$

then

$$\begin{aligned}
\langle d\mathbf{N}_\mathbf{p}(\mathbf{w}_1), d\mathbf{N}_\mathbf{p}(\mathbf{w}_2) \rangle &= \langle -k_1 a_1\mathbf{e}_1 - k_2 a_2\mathbf{e}_2, -k_1 b_1\mathbf{e}_1 - k_2 b_2\mathbf{e}_2 \rangle \\
&= k_1^2 a_1 b_1 + k_2^2 a_2 b_2 = -k_1 k_2(a_1 b_1 + a_2 b_2) \\
&= -k_1 k_2 \langle \mathbf{w}_1, \mathbf{w}_2 \rangle = -K(\mathbf{p})\langle \mathbf{w}_1, \mathbf{w}_2 \rangle.
\end{aligned}$$

Now consider the two intersecting parametrized regular curves α_1, α_2 on S such that $\alpha'_1(0) = \mathbf{w}_1$ and $\alpha'_2(0) = \mathbf{w}_2$. Note that S has no planar or parabolic points because $K(\mathbf{p}) = k_1 k_2 = -k_1^2 < 0$. By Exercise 9(a), $\mathbf{N} \circ \alpha_i$ is a parametrized regular curve on the sphere $\mathbb{S}^2$ for $i = 1, 2$. Observe that

$$(\mathbf{N} \circ \alpha_i)'(0) = d\mathbf{N}_\mathbf{p}(\alpha'_i(0)) = d\mathbf{N}_\mathbf{p}(\mathbf{w}_i)$$

and

$$\left| d\mathbf{N}_\mathbf{p}(\mathbf{w}_1) \right| = |-k_1 a_1\mathbf{e}_1 - k_2 a_2\mathbf{e}_2| = \sqrt{k_1^2 a_1^2 + k_2^2 a_2^2} = \sqrt{-k_1 k_2(a_1^2 + a_2^2)} = \sqrt{-K(\mathbf{p})}|\mathbf{w}_1|.$$

Let θ be an angle of two curves α_1 and α_2, and let φ be an angle of two curves $\mathbf{N} \circ \alpha_1$ and $\mathbf{N} \circ \alpha_2$. Then

$$\begin{aligned}
\cos\varphi &= \frac{\langle (\mathbf{N} \circ \alpha_1)'(0), (\mathbf{N} \circ \alpha_2)'(0) \rangle}{|(\mathbf{N} \circ \alpha_1)'(0)||(\mathbf{N} \circ \alpha_2)'(0)|} = \frac{\langle d\mathbf{N}_\mathbf{p}(\mathbf{w}_1), d\mathbf{N}_\mathbf{p}(\mathbf{w}_2) \rangle}{|d\mathbf{N}_\mathbf{p}(\mathbf{w}_1)||d\mathbf{N}_\mathbf{p}(\mathbf{w}_2)|} \\
&= \frac{-K(\mathbf{p})\langle \mathbf{w}_1, \mathbf{w}_2 \rangle}{\sqrt{-K(\mathbf{p})}|\mathbf{w}_1|\sqrt{-K(\mathbf{p})}|\mathbf{w}_2|} = \frac{\langle \mathbf{w}_1, \mathbf{w}_2 \rangle}{|\mathbf{w}_1||\mathbf{w}_2|} = \frac{\langle \alpha'_1(0), \alpha'_2(0) \rangle}{|\alpha'_1(0)||\alpha'_2(0)|} = \cos\theta.
\end{aligned}$$

Therefore θ and φ are equal up to a sign.

18. The normal curvatures are $\lambda_k = k_1 \cos^2\theta_k + k_2 \sin^2\theta_k$, where $\theta_k = 2\pi(k-1)/m$ for $1 \leq k \leq m$. Since

$$\sum_{k=1}^{m} \cos^2 \theta_k = \sum_{k=1}^{m} \frac{1 + \cos 2\theta_k}{2} = \frac{m}{2} + \frac{1}{2} \sum_{k=1}^{m} \cos 2\theta_k = \frac{m}{2} + \frac{1}{2} \operatorname{Re}\left(\sum_{k=1}^{m} e^{2\theta_k i} \right)$$

$$= \frac{m}{2} + \frac{1}{2} \operatorname{Re}\left(\frac{e^{2m(2\pi/m)i} - 1}{e^{2(2\pi/m)i} - 1} \right) = \frac{m}{2} + \frac{1}{2} \operatorname{Re}(0) = \frac{m}{2},$$

we obtain

$$\sum_{k=1}^{m} \lambda_k = \sum_{k=1}^{m} (k_1 \cos^2 \theta_k + k_2 \sin^2 \theta_i) = k_1 \left(\sum_{k=1}^{m} \cos^2 \theta_k \right) + k_2 \left(\sum_{k=1}^{m} \sin^2 \theta_k \right)$$

$$= k_1 \left(\frac{m}{2} \right) + k_2 \left(m - \frac{m}{2} \right) = m \cdot \frac{k_1 + k_2}{2} = mH.$$

19. a. Observe that

$$\mathbf{t} = \mathbf{e}_1 \cos \varphi + \mathbf{e}_2 \sin \varphi, \quad \mathbf{h} = \mathbf{e}_1 \cos(\varphi + \pi/2) + \mathbf{e}_2 \sin(\varphi + \pi/2) = -\mathbf{e}_1 \sin \varphi + \mathbf{e}_2 \cos \varphi.$$

Since $\mathbf{N}(s) = (\mathbf{N} \circ \alpha)(s)$ implies $\mathbf{N}'(s) = d\mathbf{N}_{\alpha(s)}(\alpha'(s))$, we have

$$\mathbf{N}'(0) = d\mathbf{N}_{\mathbf{p}}(\alpha'(0)) = d\mathbf{N}_{\mathbf{p}}(\mathbf{e}_1 \cos \varphi + \mathbf{e}_2 \sin \varphi) = -k_1 \mathbf{e}_1 \cos \varphi - k_2 \mathbf{e}_2 \sin \varphi.$$
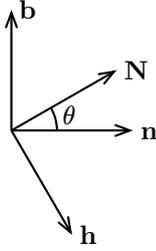
Therefore

$$\tau_g = \langle \mathbf{N}'(0), \mathbf{h} \rangle = \langle -k_1 \mathbf{e}_1 \cos \varphi - k_2 \mathbf{e}_2 \sin \varphi, -\mathbf{e}_1 \sin \varphi + \mathbf{e}_2 \cos \varphi \rangle$$

$$= k_1 \cos \varphi \sin \varphi - k_2 \cos \varphi \sin \varphi = (k_1 - k_2) \cos \varphi \sin \varphi.$$

b. Differentiating both sides of $\cos \theta = \langle \mathbf{N}, \mathbf{n} \rangle$ with respect to s,

$$-\frac{d\theta}{ds} \sin \theta = \langle \mathbf{N}', \mathbf{n} \rangle + \langle \mathbf{N}, \mathbf{n}' \rangle.$$

Since $\mathbf{N} \cdot \mathbf{N} = 1$ implies $\mathbf{N}' \cdot \mathbf{N} = 0$, we have $\mathbf{N}' \in T_{\mathbf{p}}(S)$. Hence there exist $a, b \in \mathbb{R}$ such that $\mathbf{N}' = a\mathbf{t} + b\mathbf{h}$. Since $\langle \mathbf{N}', \mathbf{t} \rangle = k_n$ and $\langle \mathbf{N}', \mathbf{h} \rangle = \tau_g$, we obtain $\mathbf{N}' = k_n \mathbf{t} + \tau_g \mathbf{h}$. Moreover, Since θ is the angle between $\mathbf{n}$ and $\mathbf{N}$, we can write $\mathbf{N}$ and $\mathbf{h}$ as

$$\mathbf{N} = \mathbf{n} \cos \theta + \mathbf{b} \sin \theta, \quad \mathbf{h} = \mathbf{n} \sin \theta - \mathbf{b} \cos \theta.$$

Note that four vectors $\mathbf{n}, \mathbf{b}, \mathbf{N}, \mathbf{h}$ are in the same plane (perpendicular to $\mathbf{t}$). Therefore

$$-\frac{d\theta}{ds} \sin \theta = \langle \mathbf{N}', \mathbf{n} \rangle + \langle \mathbf{N}, \mathbf{n}' \rangle = \langle -k_n \mathbf{t} + \tau_g \mathbf{h}, \mathbf{n} \rangle + \langle \mathbf{N}, -k\mathbf{t} - \tau \mathbf{b} \rangle$$

$$= \tau_g \langle \mathbf{h}, \mathbf{n} \rangle - \tau \langle \mathbf{N}, \mathbf{b} \rangle = \tau_g \sin \theta - \tau \sin \theta.$$

Thus we obtain $d\theta/ds = \tau - \tau_g$.

c. First suppose that C is a line of curvature of S. Then there exists differentiable function $\lambda(s)$ such that $\mathbf{N}'(s) = \lambda(s)\alpha'(s) = \lambda(s)\mathbf{t}(s)$, by Prop. 3. Hence $\tau_g = \langle \mathbf{N}', \mathbf{h} \rangle = \langle \lambda \mathbf{t}, \mathbf{h} \rangle \equiv 0$ and C has geodesic torsion identically zero.

Conversely, suppose that C has geodesic torsion identically zero. Then $\mathbf{N}'(s)$ is perpendicular to $\mathbf{N}(s)$ and $\mathbf{h}(s)$ for each s, so $\mathbf{N}'(s)$ is parallel to $\mathbf{h}(s) \wedge \mathbf{N}(s) = \mathbf{t}(s)$. Hence there exists differentiable function $\lambda(s)$ such that $\mathbf{N}'(s) = \lambda(s)\mathbf{t}(s)$, and we conclude that C is line of curvature by Prop. 3.

20. Let S_1, S_2, and S_3 be the surfaces that pass through $\mathbf{p}$. We can assume, changing the direction of $\mathbf{N}_3$ if necessary, that $\{\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3\}$ constitute the positive orthonormal basis for $\mathbb{R}^3$. We assert that the

geodesic torsions τ_1, τ_1' of $C_1 = S_2 \cap S_3$ relative to S_2 and S_3 are equal. Since the tangent vector $\mathbf{t}_1$ of C_1 is perpendicular to both $\mathbf{N}_2$ and $\mathbf{N}_3$, we have $\mathbf{t}_1 = \pm \mathbf{N}_1$. Then by definition

$$\tau_1 = \langle \mathbf{N}_2', \mathbf{N}_2 \wedge \mathbf{t}_1 \rangle = \langle \mathbf{N}_2', \mp \mathbf{N}_3 \rangle = -\langle d\mathbf{N}_2(\mathbf{N}_1), \mathbf{N}_3 \rangle,$$

$$\tau_1' = \langle \mathbf{N}_3', \mathbf{N}_3 \wedge \mathbf{t}_1 \rangle = \langle \mathbf{N}_3', \pm \mathbf{N}_2 \rangle = \langle d\mathbf{N}_3(\mathbf{N}_1), \mathbf{N}_2 \rangle.$$

Since $\langle \mathbf{N}_2, \mathbf{N}_3 \rangle = 0$, we obtain $\langle \mathbf{N}_2', \mathbf{N}_3 \rangle + \langle \mathbf{N}_2, \mathbf{N}_3' \rangle = 0$. Therefore $\tau_1 = \tau_1'$.

Now let τ_2 be the geodesic torsion of $C_2 = S_1 \cap S_3$ relative to S_1 or S_3, and let τ_3 be the geodesic torsion of $C_3 = S_1 \cap S_2$ relative to S_1 or S_2. Observe that

$$\tau_2 = \langle d\mathbf{N}_1(\mathbf{N}_2), \mathbf{N}_3 \rangle = -\langle d\mathbf{N}_3(\mathbf{N}_2), \mathbf{N}_1 \rangle.$$

Therefore $\tau_1 + \tau_2 = -\langle d\mathbf{N}_2(\mathbf{N}_1), \mathbf{N}_3 \rangle + \langle d\mathbf{N}_1(\mathbf{N}_2), \mathbf{N}_3 \rangle = 0$. Similarly, $\tau_2 + \tau_3 = \tau_3 + \tau_1 = 0$ implies that $\tau_1 = \tau_2 = \tau_3 = 0$. Since the lines having geodesic torsion identically zero are lines of curvature, C_1, C_2 and C_3 are the lines of curvature.

> **Unsolved** It is not immediately obvious how to establish that $\langle d\mathbf{N}_2(\mathbf{N}_1), \mathbf{N}_3 \rangle - \langle d\mathbf{N}_1(\mathbf{N}_2), \mathbf{N}_3 \rangle = 0$ at $\mathbf{p}$.

3-3. The Gauss Map in Local Coordinates

1. Let $h(x, y) = axy$, so the parametrization of the hyperboloid is $\mathbf{x}(x, y) = (x, y, h(x, y))$. Then

$$h_x = ay = 0, \quad h_y = ax = 0, \quad h_{xx} = 0, \quad h_{xy} = a, \quad h_{yy} = 0$$

at $(0, 0, 0)$. Thus by Example 5,

$$K = \frac{h_{xx} h_{yy} - h_{xy}^2}{\left(1 + h_x^2 + h_y^2\right)^2} = \frac{0 \cdot 0 - a^2}{\left(1 + 0^2 + 0^2\right)^2} = -a^2, \quad H = \frac{\left(1 + h_x^2\right) h_{yy} - 2 h_x h_y h_{xy} + \left(1 + h_y^2\right) h_{xx}}{2 \left(1 + h_x^2 + h_y^2\right)^{3/2}} = 0.$$

2. Let $\mathbf{x}(u, v) = (v \cos u, v \sin u, cu)$ be the parametrization of the helicoid. Then

$$\mathbf{x}_u = (-v \sin u, v \cos u, c), \ \mathbf{x}_v = (\cos u, \sin u, 0) \implies E = v^2 + c^2, \ F = 0, \ G = 1.$$

Furthermore,

$$\mathbf{x}_{uu} = (-v \cos u, -v \sin u, 0), \ \mathbf{x}_{uv} = (-\sin u, \cos u, 0), \ \mathbf{x}_{vv} = (0, 0, 0)$$

implies that

$$e = \frac{(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uu})}{\sqrt{v^2 + c^2}} = 0, \ f = \frac{(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uv})}{\sqrt{v^2 + c^2}} = \frac{c}{\sqrt{v^2 + c^2}}, \ g = \frac{(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{vv})}{\sqrt{v^2 + c^2}} = 0.$$

Since $e = g = 0$, the asymptotic curves are the coordinate lines. The differential equation of the lines of curvature is

$$\left| \begin{pmatrix} (v')^2 & -u'v' & (u')^2 \\ E & F & G \\ e & f & g \end{pmatrix} \right| = \left| \begin{pmatrix} (v')^2 & -u'v' & (u')^2 \\ v^2 + c^2 & 0 & 1 \\ 0 & \frac{c}{\sqrt{v^2+c^2}} & 0 \end{pmatrix} \right| = \left(-\frac{c}{\sqrt{v^2 + c^2}} \right)(v')^2 + c\sqrt{v^2 + c^2}(u')^2 = 0.$$

Consequently, the lines of curvature are given by

$$(u')^2 = \frac{(v')^2}{v^2 + c^2} \implies u' = \pm \frac{v'}{\sqrt{v^2 + c^2}} \implies u = \pm \log\left(v + \sqrt{v^2 + c^2}\right) + C$$

where C is a constant. Finally, the mean curvature is

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)} = 0.$$

3. Computing the derivatives of $\mathbf{x}$, we obtain

$$\mathbf{x}_u = (-\cosh v \sin u, \cosh v \cos u, 0), \quad \mathbf{x}_v = (\sinh v \cos u, \sinh v \sin u, 1)$$

thus the coefficients of the first fundamental form are

$$E = \cosh^2 v, \quad F = 0, \quad G = \sinh^2 v + 1 = \cosh^2 v.$$

Since

$$\mathbf{x}_{uu} = (-\cosh v \cos u, -\cosh v \sin u, 0),$$
$$\mathbf{x}_{uv} = (-\sinh v, \sin u, \sinh v \cos u, 0),$$
$$\mathbf{x}_{vv} = (\cosh v \cos u, \cosh v \sin u, 0),$$

the coefficients of the second fundamental form are

$$e = \frac{(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uu})}{\cosh^2 v} = -1, \ f = \frac{(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uv})}{\cosh^2 v} = 0, \ g = \frac{(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{vv})}{\cosh^2 v} = 1.$$

The differential equation of asymptotic curves is

$$e(u')^2 + 2fu'v' + g(v')^2 = 0 \implies -(u')^2 + (v')^2 = 0 \implies u' \pm v' = 0.$$

Therefore the asymptotic curves are given by $u \pm v = \text{const.}$.

4. Let $\mathbf{x}(u,v) = (u, v, uv)$ be the parametrization of the curve $z = xy$. Computing the derivatives,

$$\mathbf{x}_u = (1, 0, v), \ \mathbf{x}_v = (0, 1, u), \ \mathbf{x}_{uu} = (0,0,0), \ \mathbf{x}_{uv} = (0,0,1), \ \mathbf{x}_{vv} = (0,0,0).$$

Therefore the coefficients of the fundamental forms are

$$E = 1 + v^2, \ F = uv, \ G = 1 + u^2, \ e = 0, \ f = \frac{1}{\sqrt{1+u^2+v^2}}, \ g = 0.$$

Since $e = g = 0$, the asymptotic curves are the coordinate curves $u = \text{const.}, v = \text{const.}$ of the parametrization. The differential equation of the lines of curvature is

$$\left| \begin{pmatrix} (v')^2 & -u'v' & (u')^2 \\ E & F & G \\ e & f & g \end{pmatrix} \right| = \left| \begin{pmatrix} (v')^2 & -u'v' & (u')^2 \\ 1+v^2 & uv & 1+u^2 \\ 0 & \frac{1}{\sqrt{1+u^2+v^2}} & 0 \end{pmatrix} \right|$$

$$= -\frac{1+u^2}{\sqrt{1+u^2+v^2}}(v')^2 + \frac{1+v^2}{\sqrt{1+u^2+v^2}}(u')^2$$

$$= 0.$$

Then

$$(1+v^2)(u')^2 = (1+u^2)(v')^2 \implies \frac{(u')^2}{1+u^2} = \frac{(v')^2}{1+v^2} \implies \frac{u'}{\sqrt{1+u^2}} = \pm\frac{v'}{\sqrt{1+v^2}},$$

thus the lines of curvature is given by $\log\left(u + \sqrt{1+u^2}\right) = \pm\log\left(v + \sqrt{1+v^2}\right) + \text{const.}$.

5. **a.** $\mathbf{x}_u = (1 - u^2 + v^2, 2vu, 2u), \ \mathbf{x}_v = (2uv, 1 - v^2 + u^2, -2v)$. Hence

$$E = (1 - u^2 + v^2)^2 + (2vu)^2 + (2u)^2 = (1 + u^2 + v^2)^2,$$
$$F = (1 - u^2 + v^2)(2uv) + (2vu)(1 - v^2 + u^2) + (2u)(-2v) = 0,$$
$$G = (2uv)^2 + (1 - v^2 + u^2)^2 + (-2v)^2 = (1 + u^2 + v^2)^2.$$

b. $\mathbf{x}_{uu} = (-2u, 2v, 2), \ \mathbf{x}_{uv} = (2v, 2u, 0), \ \mathbf{x}_{vv} = (2u, -2v, -2)$. Thus

$$e = \frac{(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uu})}{(1 + u^2 + v^2)^2} = \frac{1}{(1 + u^2 + v^2)^2}\left| \begin{pmatrix} 1 - u^2 + v^2 & 2vu & 2u \\ 2uv & 1 - v^2 + u^2 & -2v \\ -2u & 2v & 2 \end{pmatrix} \right| = 2,$$

$$f = \frac{(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uv})}{(1 + u^2 + v^2)^2} = \frac{1}{(1 + u^2 + v^2)^2}\left| \begin{pmatrix} 1 - u^2 + v^2 & 2vu & 2u \\ 2uv & 1 - v^2 + u^2 & -2v \\ 2v & 2u & 0 \end{pmatrix} \right| = 0,$$

$$g = \frac{(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{vv})}{(1 + u^2 + v^2)^2} = \frac{1}{(1 + u^2 + v^2)^2}\left| \begin{pmatrix} 1 - u^2 + v^2 & 2vu & 2u \\ 2uv & 1 - v^2 + u^2 & -2v \\ 2u & -2v & -2 \end{pmatrix} \right| = -2.$$

c. Observe that

$$K = \frac{eg - f^2}{EG - F^2} = -\frac{4}{(1 + u^2 + v^2)^4},$$

$$H = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2} = \frac{1}{2}\frac{2(1 + u^2 + v^2)^2 - 2(1 + u^2 + v^2)^2}{(1 + u^2 + v^2)^4} = 0.$$

Since the principal curvatures k_1, k_2 are the roots of the quadratic equation

$$k^2 - 2Hk + K = k^2 - \frac{4}{(1 + u^2 + v^2)^4} = 0,$$

we obtain

$$k_1 = \frac{2}{(1 + u^2 + v^2)^2}, \quad k_2 = -\frac{2}{(1 + u^2 + v^2)^2}$$

d. Since $f = F = 0$, the lines of curvature are the coordinate curves.

e. The differential equation of the asymptotic curves is

$$e(u')^2 + 2fu'v' + g(v')^2 = 0 \implies (u')^2 - (v')^2 \implies u' \pm v' = 0.$$

Therefore the asymptotic curves are $u + v = \text{const.}$ and $u - v = \text{const.}$.

6. a. By taking the line r as the z axis and a normal to r as the x axis, then

$$\frac{dz}{dx} = -\frac{\sqrt{1 - x^2}}{x}.$$

By setting $x = \sin t$, for $\pi/2 < t < \pi$, we obtain

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} = \frac{\cos t}{\sin t} \cdot \cos t = \frac{\cos^2 t}{\sin t}.$$

Therefore

$$z = \int \frac{\cos^2 t}{\sin t}\, dt = \cos t + \log \tan \frac{t}{2} + C.$$

If $z(\pi/2) = 0$, then $C = 0$. Note that if $z < 0$, we set $x = \sin t$ for $0 < t < \pi/2$, then the differential equation is given by

$$\frac{dx}{dz} = \frac{\sqrt{1 - x^2}}{x} \implies \frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} = \frac{\cos t}{\sin t} \cdot \cos t = \frac{\cos^2 t}{\sin t}.$$

Therefore we obtain $z = \cos t + \log \tan t/2$ again.
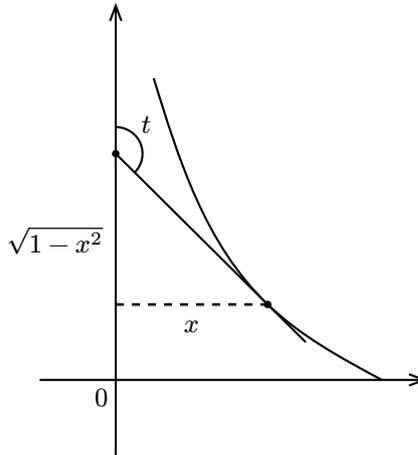

b. Define $\mathbf{x} : U \to S$ as

$$\mathbf{x}(u, v) = \left(\sin u \cos v, \sin u \sin v, \cos u + \log \tan \frac{u}{2}\right),$$

where $U = \{(u, v) \in \mathbb{R}^2 \mid 0 < u < \pi,\ 0 < v < 2\pi\}$. Observe that the regions determined by $0 < u < \pi/2$ and $\pi/2 < u < \pi$ are regular surface because $\sin u > 0$ and $\cos u + \log \tan u/2$ are differentiable. However, the points in a circle $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ z = 0\}$ given by $u = \pi/2$ is not regular, since

$$\mathbf{x}_u = \left(\cos u \cos v, \cos u \sin v, \frac{\cos^2 u}{\sin u} \right) = (0, 0, 0),$$

$$\mathbf{x}_v = (-\sin u \sin v, \sin u \cos v, 0) = (-\sin v, \cos v, 0)$$

so $d\mathbf{x}_{\mathbf{q}}$ is not one-to-one.

c. Since

$$\mathbf{x}_u = \left(\cos u \cos v, \cos u \sin v, \frac{\cos^2 u}{\sin u} \right), \quad \mathbf{x}_v = (-\sin u \sin v, \sin u \cos v, 0),$$

the coefficients of the first fundamental form are

$$E = \cos^2 u + \frac{\cos^4 u}{\sin^2 u} = \frac{\cos^2 u}{\sin^2 u}, \quad F = 0, \quad G = \sin^2 u.$$

The second derivatives of $\mathbf{x}$ are

$$\mathbf{x}_{uu} = \left(-\sin u \cos v, -\sin u \sin v, -2\cos u - \frac{\cos^3 u}{\sin^2 u} \right),$$

$$\mathbf{x}_{uv} = (-\cos u \sin v, \cos u \cos v, 0),$$

$$\mathbf{x}_{vv} = (-\sin u \cos v, -\sin u \sin v, 0).$$

Since $\sqrt{EG - F^2} = |\cos u|$, the coefficients of the second fundamental form are

$$e = -\frac{\cos^2 u}{|\cos u| \sin u}, \quad f = 0, \quad g = \frac{\cos^2 u \sin u}{|\cos u|}.$$

Therefore the Gaussian curvature of a regular point is

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-(\cos^4 u)/|\cos u|^2}{\cos^2 u} \equiv -1.$$

7. a. By Example 4, the Gaussian curvature is given by $K = -\varphi''/\varphi$. Thus $\varphi'' + K\varphi = 0$. Since

$$(\varphi')^2 + (\psi')^2 = 1,$$

we obtain $\psi' = \sqrt{1 - (\varphi')^2}$ and $\varphi = \int \sqrt{1 - (\varphi')^2}\, dv$.

b. By setting $K = 1$, the differential equation is given by $\varphi'' + \varphi = 0$, which implies

$$\varphi(v) = A \sin v + C \cos v.$$

To intersect the plane xOy perpendicularly, it must be $\varphi'(0) = A = 0$. Therefore we obtain

$$\varphi(v) = C \cos v, \quad \psi(v) = \int_0^v \sqrt{1 - (\varphi')^2}\, dv = \int_0^v \sqrt{1 - C^2 \sin^2 v}\, dv.$$

This integral makes sense only if $1 - C^2 \sin^2 v \geq 0$. Therefore the domain of v is

$$|v| \leq \arcsin \frac{1}{|C|}.$$

c. By setting $K = -1$, the differential equation becomes $\varphi'' - \varphi = 0$, whose general solution is

$$\varphi(v) = A \cosh v + B \sinh v.$$

If $(A, B) = (C, 0)$, then the surface is given by

$$\varphi(v) = C \cosh v, \quad \psi(v) = \int_0^v \sqrt{1 - C^2 \sinh^2 v}\, dv \quad \text{(Case 1)}.$$

If $(A, B) = (0, C)$, then the surface is given by

$$\varphi(v) = C \cosh v, \quad \psi(v) = \int_0^v \sqrt{1 - C^2 \cosh^2 v}\, dv \quad \text{(Case 2)}.$$

If $(A, B) = (1, 1)$, then the surface is given by

$$\varphi(v) = e^v, \quad \psi(v) = \int_0^v \sqrt{1 - e^2 v}\, dv \quad \text{(the case 3)}.$$

d. Substitute $e^v = \sin t$. Then $e^v\, dv = \cos t\, dt$, thus $\varphi(v) = e^v = \sin t$ and

$$\psi(v) = \int_0^v \sqrt{1 - e^{2v}}\, dv = \int_0^t \sqrt{1 - \sin^2 t}\, \frac{\cos t}{\sin t}\, dt = \int_0^t \frac{\cos^2 t}{\sin t}\, dt = \cos t + \log \tan \frac{t}{2}.$$

Therefore the surface of type 3 is the pseudosphere of Exercise 6.

e. By setting $K = 0$, the differential equation becomes $\varphi'' = 0$ and the solution is

$$\varphi(v) = Av + B, \quad \psi(v) = \int_0^v \sqrt{1 - A^2}\, dv = \sqrt{1 - A^2}\, v.$$

If $A^2 \neq 0, 1$, then the surface is the right circular cone. If $A = 0$ and $B \neq 0$, then the surface is the right circular cylinder. If $A^2 = 1$, then $\psi(v)$ is constant, so the surface is the plane. Note that if $A = B = 0$, then $\varphi(v) \equiv 0$, so it cannot be a generating curve.

8. a. Since S and $\overline{S}$ have contact of order ≥ 2, by definition, there exist parametrizations $\mathbf{y}(\overline{u}, \overline{v}), \overline{\mathbf{y}}(\overline{u}, \overline{v})$ of S and $\overline{S}$, respectively, such that

$$\mathbf{y}_{\overline{u}} = \overline{\mathbf{y}}_{\overline{u}}, \quad \mathbf{y}_{\overline{v}} = \overline{\mathbf{y}}_{\overline{v}}, \quad \mathbf{y}_{\overline{u}\overline{u}} = \overline{\mathbf{y}}_{\overline{u}\overline{u}}, \quad \mathbf{y}_{\overline{u}\overline{v}} = \overline{\mathbf{y}}_{\overline{u}\overline{v}}, \quad \mathbf{y}_{\overline{v}\overline{v}} = \overline{\mathbf{y}}_{\overline{v}\overline{v}},$$

Now let $\mathbf{h}$ be the change of variables, that is, $\mathbf{h}(u, v) = (\overline{u}(u, v), \overline{v}(v, v))$. Then $\mathbf{h}$ is a diffeomorphism and we have $\mathbf{x} = \mathbf{y} \circ \mathbf{h}$ and $\overline{\mathbf{x}} = \overline{\mathbf{y}} \circ \mathbf{h}$. Then

$$\mathbf{x}_u = \mathbf{y}_{\overline{u}} \frac{\partial \overline{u}}{\partial u} + \mathbf{y}_{\overline{v}} \frac{\partial \overline{v}}{\partial u} = \overline{\mathbf{y}}_{\overline{u}} \frac{\partial \overline{u}}{\partial u} + \overline{\mathbf{y}}_{\overline{v}} \frac{\partial \overline{v}}{\partial u} = \overline{\mathbf{x}}_u, \quad \mathbf{x}_v = \mathbf{y}_{\overline{u}} \frac{\partial \overline{u}}{\partial v} + \mathbf{y}_{\overline{v}} \frac{\partial \overline{v}}{\partial v} = \overline{\mathbf{y}}_{\overline{u}} \frac{\partial \overline{u}}{\partial v} + \overline{\mathbf{y}}_{\overline{v}} \frac{\partial \overline{v}}{\partial v} = \overline{\mathbf{x}}_v,$$

$$\mathbf{x}_{uu} = \left(\mathbf{y}_{\overline{u}\overline{u}} \frac{\partial \overline{u}}{\partial u} + \mathbf{y}_{\overline{u}\overline{v}} \frac{\partial \overline{v}}{\partial u} \right) \frac{\partial \overline{u}}{\partial u} + \mathbf{y}_{\overline{u}} \frac{\partial^2 \overline{u}}{\partial u^2} + \left(\mathbf{y}_{\overline{v}\overline{u}} \frac{\partial \overline{u}}{\partial u} + \mathbf{y}_{\overline{v}\overline{v}} \frac{\partial \overline{v}}{\partial u} \right) \frac{\partial \overline{v}}{\partial u} + \mathbf{y}_{\overline{v}} \frac{\partial^2 \overline{v}}{\partial u^2} = \overline{\mathbf{x}}_{uu}.$$

Therefore we conclude that $\mathbf{x}$ and $\overline{\mathbf{x}}$ have contact of order ≥ 2. Observe that

$$(f \circ \overline{\mathbf{x}})_u = f(\overline{x}(u, v), \overline{y}(u, v), \overline{z}(u, v))_u = f_x \overline{x}_u + f_y \overline{y}_u + f_z \overline{z}_u = f_x x_u + f_y y_u + f_z z_u = (f \circ \mathbf{x})_u$$

and

$$(f \circ \overline{\mathbf{x}})_{uu} = (f_{xx} \overline{x}_u + f_{xy} \overline{y}_u + f_{xz} \overline{z}_u) \overline{x}_u + f_x \overline{x}_{uu} + (f_{yx} \overline{x}_u + f_{yy} \overline{y}_u + f_{yz} \overline{z}_u) \overline{y}_u + f_y \overline{y}_{uu}$$

$$+ (f_{zx} \overline{x}_u + f_{zy} \overline{y}_u + f_{zz} \overline{z}_u) \overline{z}_u + f_z \overline{z}_{uu}$$

$$= (f_{xx} \overline{x}_u + f_{xy} \overline{y}_u + f_{xz} \overline{z}_u) \overline{x}_u + f_x \overline{x}_{uu} + (f_{yx} \overline{x}_u + f_{yy} \overline{y}_u + f_{yz} \overline{z}_u) \overline{y}_u + f_y \overline{y}_{uu}$$

$$+ (f_{zx} \overline{x}_u + f_{zy} \overline{y}_u + f_{zz} \overline{z}_u) \overline{z}_u + f_z \overline{z}_{uu}$$

$$= (f \circ \mathbf{x})_{uu},$$

Hence the partial derivatives of order ≤ 2 of $(f \circ \overline{\mathbf{x}})$ and $(f \circ \mathbf{x})$ coincide, and we conclude that they become zero at the same time.

b. Let $\mathbf{x}(x, y) = (x, y, f(x, y))$ and $\overline{\mathbf{x}}(x, y) = \left(x, y, \overline{f}(x, y) \right)$ be the parametrizations at $\mathbf{p}$ for S and $\overline{S}$, respectively. We define a function $h(x, y, z) = f(x, y) - z$. Since $h \circ \mathbf{x} = 0$ and $h \circ \overline{\mathbf{x}} = f - \overline{f}$, by (a) applied to the function h, $f - \overline{f}$ has partial derivatives of order ≤ 2 equal to zero at $(0, 0)$.

c. Let $\mathbf{x}(x, y) = (x, y, f(x, y))$ be the parametrization for S at $(0, 0, 0)$, and let

$$\overline{\mathbf{x}}(x, y) = \left(x, y, \frac{1}{2}(x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}) \right)$$

be the paraboloid. Since

$$\mathbf{x}_x = \overline{\mathbf{x}}_x = (1, 0, 0), \quad \mathbf{x}_y = \overline{\mathbf{x}}_y = (0, 1, 0),$$

$$\mathbf{x}_{xx} = \overline{\mathbf{x}}_{xx} = (0, 0, f_{xx}), \quad \mathbf{x}_{xy} = \overline{\mathbf{x}}_{xy} = (0, 0, f_{xy}), \quad \mathbf{x}_{yy} = \overline{\mathbf{x}}_{yy} = (0, 0, f_{yy}),$$

we conclude that two curves have contact of order ≥ 2.

d. Since contact of order ≥ 2 implies contact of order ≥ 1, the paraboloid passes through $\mathbf{p}$ and is tangent to the surface at $\mathbf{p}$. By taking the plane $T_{\mathbf{p}}(S)$ as the xy plane, the equation of the paraboloid becomes

$$\overline{f}(x, y) = ax^2 + 2bxy + cy^2 + dx + ey.$$

Let $z = f(x, y)$ be the representation of the surface in the plane $T_{\mathbf{p}}(S)$. By using part (b), the function $f - \overline{f}$ has all partial derivatives of order ≤ 2 equal to zero. Hence

$$a = \frac{1}{2}f_{xx}, \quad b = f_{xy}, \quad c = \frac{1}{2}f_{yy}, \quad d = e = 0.$$

e. Suppose that the surfaces S and $\overline{S}$ have contact of order ≥ 2 at $\mathbf{p}$. Since two curves have same tangent plane at $\mathbf{p}$, we can take the plane $T_{\mathbf{p}}(S) = T_{\mathbf{p}}(\overline{S})$ as the xy plane. Let $z = f(x, y)$ and $z = \overline{f}(x, y)$ be the representations of the surfaces S and $\overline{S}$, respectively, in the plane $T_{\mathbf{p}}(S)$. Then by (b), $f(x, y) - \overline{f}(x, y)$ has all partial derivatives of order ≤ 2 equal to zero. Therefore we conclude that the osculating paraboloids

$$z = \frac{1}{2}\left(x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}\right), \quad z = \frac{1}{2}\left(x^2 \overline{f}_{xx} + 2xy \overline{f}_{xy} + y^2 \overline{f}_{yy}\right)$$

coincide. Furthermore, since the Gaussian and mean curvatures can be represented by the partial derivatives of order ≤ 2 of f and $\overline{f}$, the Gaussian and mean curvatures of S and $\overline{S}$ at $\mathbf{p}$ are equal.

f. In Exercise 27(d) of Sec. 2-4, we showed that if S and $\overline{S}$ are regular surfaces which have contact of order ≥ 1 at $\mathbf{p}$ and $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism, then $\varphi(S)$ and $\varphi(\overline{S})$ are regular surfaces which have contact of order ≥ 1 at $\varphi(\mathbf{p})$. Hence it is sufficient to show that if S and $\overline{S}$ have contact of order ≥ 2, then $\varphi(S)$ and $\varphi(\overline{S})$ have contact of order ≥ 2 at $\varphi(\mathbf{p})$.

Since S and $\overline{S}$ have contact of order ≥ 2 at $\mathbf{p}$, there exist parametrizations $\mathbf{x}(u, v)$, $\overline{\mathbf{x}}(u, v)$ of S and $\overline{S}$, respectively, such that

$$\mathbf{x}_u = \overline{\mathbf{x}}_u, \quad \mathbf{x}_v = \overline{\mathbf{x}}_v, \quad \mathbf{x}_{uu} = \overline{\mathbf{x}}_{uu}, \quad \mathbf{x}_{uv} = \overline{\mathbf{x}}_{uv}, \quad \mathbf{x}_{vv} = \overline{\mathbf{x}}_{vv}.$$

We showed that $(\varphi \circ \mathbf{x})(u, v)$, $(\varphi \circ \overline{\mathbf{x}})(u, v)$ is parametrizations for $\mathbf{F}(S)$ and $\mathbf{F}(\overline{S})$ at $\mathbf{F}(\mathbf{p})$. Furthermore, the partial derivatives of order ≤ 2 of $\varphi \circ \mathbf{x}$ consists of the partial derivatives of order ≤ 2 of φ and $\mathbf{x}$, hence we obtain

$$(f \circ \mathbf{x})_u = (f \circ \overline{\mathbf{x}})_u, \quad (f \circ \mathbf{x})_v = (f \circ \overline{\mathbf{x}})_v,$$

$$(f \circ \mathbf{x})_{uu} = (f \circ \overline{\mathbf{x}})_{uu}, \quad (f \circ \mathbf{x})_{uv} = (f \circ \overline{\mathbf{x}})_{uv}, \quad (f \circ \mathbf{x})_{vv} = (f \circ \overline{\mathbf{x}})_{vv}.$$

Hence $\varphi(S)$ and $\varphi(\overline{S})$ are regular surfaces which have contact of order ≥ 2 at $\varphi(\mathbf{p})$.

g. Let $\mathbf{x}(u, v)$ and $\overline{\mathbf{x}}(u, v)$ be the parametrizations at $\mathbf{p}$ for S and $\overline{S}$, respectively, with

$$\mathbf{x}_u = \overline{\mathbf{x}}_u, \quad \mathbf{x}_v = \overline{\mathbf{x}}_v, \quad \mathbf{x}_{uu} = \overline{\mathbf{x}}_{uu}, \quad \mathbf{x}_{uv} = \overline{\mathbf{x}}_{uv}, \quad \mathbf{x}_{vv} = \overline{\mathbf{x}}_{vv}$$

and $\mathbf{x}(0, 0) = \overline{\mathbf{x}}(0, 0) = \mathbf{p}$. Since $\mathbf{x}$ and $\overline{\mathbf{x}}$ are differentiable, by Taylor's formula,

$$\mathbf{x}(u, v) = \mathbf{x}(0, 0) + \mathbf{x}_u u + \mathbf{x}_v v + \frac{1}{2}\left(\mathbf{x}_{uu} u^2 + 2\mathbf{x}_{uv} uv + \mathbf{x}_{vv} v^2\right) + \mathbf{R},$$

$$\overline{\mathbf{x}}(u, v) = \overline{\mathbf{x}}(0, 0) + \overline{\mathbf{x}}_u u + \overline{\mathbf{x}}_v v + \frac{1}{2}\left(\overline{\mathbf{x}}_{uu} u^2 + 2\overline{\mathbf{x}}_{uv} uv + \overline{\mathbf{x}}_{vv} v^2\right) + \overline{\mathbf{R}},$$

where the remainder $\mathbf{R}$, $\overline{\mathbf{R}}$ satisfies the condition

$$\lim_{r \to 0} \frac{\mathbf{R}}{r^2} = \lim_{r \to 0} \frac{\overline{\mathbf{R}}}{r^2} = \mathbf{0}, \quad r = \sqrt{u^2 + v^2}.$$

Let $\mathbf{N}$ be a common normal of the surfaces at $\mathbf{p}$, then

$$d = \langle \mathbf{x}(u, v) - \overline{\mathbf{x}}(u, v), \mathbf{N} \rangle = \langle \mathbf{R} - \overline{\mathbf{R}}, \mathbf{N} \rangle.$$

Therefore we have $d/r^2 = \langle \mathbf{R}/r^2 - \overline{\mathbf{R}}/r^2, \mathbf{N} \rangle \to \langle \mathbf{0}, \mathbf{N} \rangle = 0$ as $r \to 0$.

9. Two regular curves C and $\overline{C}$, with a common point $\mathbf{p}$, have contact of order $\geq n$ at $\mathbf{p}$ if there exist parametrizations $\alpha(t)$ and $\beta(t)$ in $\mathbf{p}$ of C and $\overline{C}$, respectively, such that

$$\alpha'(t) = \beta'(t), \ \alpha''(t) = \beta''(t), \ ..., \alpha^{(n)}(t) = \beta^{(n)}(t)$$

at **p**.

a. Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be a diffeomorphism. First we assert that if C is a regular curve, then $\varphi(C)$ is also a regular curve. Given point $\mathbf{p} \in C$, let $\alpha : I \to C$ be a parametrization for C at $\mathbf{p}$. We show that $\varphi \circ \alpha : I \to \varphi(C)$ is a parametrization for $\varphi(C)$ at $\varphi(\mathbf{p})$. (i, ii) $\varphi \circ \alpha$ is a differentiable homeomorphism since φ and α are differentiable and homeomorphic. (iii) Since

$$d(\varphi \circ \alpha)_t = d\varphi_{\alpha(t)} \circ d\alpha_t$$

is a composition of two injective maps $d\varphi_{\alpha(t)}$ and $d\alpha_t$, it is also one-to-one for every $t \in I$. Therefore we conclude that $\varphi(C)$ is a regular curve.

Now suppose that two regular curves C and $\overline{C}$ have contact of order $\geq n$ at **p**. Then there exist the parametrizations $\alpha(t)$ and $\beta(t)$ at **p** of C and $\overline{C}$, respectively, such that

$$\alpha'(t) = \beta'(t), \ \alpha''(t) = \beta''(t), \ ..., \alpha^{(n)}(t) = \beta^{(n)}(t)$$

at **p**. We must show that

$$(\varphi \circ \alpha)'(t) = (\varphi \circ \beta)'(t), \ (\varphi \circ \alpha)''(t) = (\varphi \circ \beta)''(t), \ ..., (\varphi \circ \alpha)^{(n)}(t) = (\varphi \circ \beta)^{(n)}(t).$$

Since $(\varphi \circ \alpha)^{(m)}(t)$, where $1 \leq m \leq n$, consists of partial derivatives of order $\leq m$ of φ and α, it follows that equation above holds. Hence $\varphi(C)$ and $\varphi\left(\overline{C}\right)$ have contact of order $\geq n$ at $\varphi(\mathbf{p})$. Therefore the notion of contact of order $\geq n$ is invariant by diffeomorphism.

b. First suppose that two curves C and $\overline{C}$ have contact of order ≥ 1 at **p**. Then there exist the parametrizations $\alpha(t)$ and $\beta(t)$ at **p** of C and $\overline{C}$, respectively, such that $\alpha'(t) = \beta'(t)$ at **p**. It follows that two curves are tangent at **p**.

Conversely, suppose that two curves C and $\overline{C}$ are tangent at **p**. Let $\alpha(t)$ and $\beta(u)$ be parametrizations at **p** of C and $\overline{C}$, respectively, such that $\alpha(t_0) = \beta(u_0) = \mathbf{p}$. Since two curves are tangent at **p**, we have $\alpha'(t_0) \parallel \beta'(u_0)$, that is, $\alpha'(t_0) = c\beta'(u_0)$ for nonzero constant c at **p**. Now we reparametrize β by $t = u/c$. Then $\tilde{\beta}(t) = \beta(u)$ satisfies the condition

$$\tilde{\beta}'(t_0) = c\beta'(u_0) = \alpha'(t_0).$$

Therefore we conclude that two curves have contact of order ≥ 1 at **p**.

10. a. First suppose that C and S have contact of order $\geq n$. Then there exists a curve $\overline{C} \subset S$ passing through **p** such that C and $\overline{C}$ have contact of order $\geq n$. Let $\alpha(t)$ and $\overline{\alpha}(t)$ be the parametrizations at **p** of C and $\overline{C}$, respectively, such that

$$\alpha(0) = \overline{\alpha}(0) = \mathbf{p}, \ \alpha'(0) = \overline{\alpha}'(0), \ \alpha''(0) = \overline{\alpha}''(0), \ ..., \alpha^{(n)}(0) = \overline{\alpha}^{(n)}(t).$$

$f(x(0), y(0), z(0)) = f(\alpha(0)) = \mathbf{p}$ because $\mathbf{p} \in S$. Since $\overline{C} \subset S$, we have $f(\overline{\alpha}(t)) = 0$. Hence

$$\frac{d}{dt} f(\overline{\alpha}(t)) = 0, \ \frac{d^2}{dt^2} f(\overline{\alpha}(t)) = 0, \ ..., \ \frac{d^n}{dt^n} f(\overline{\alpha}(t)) = 0$$

at $t = 0$. Note that for every $m \leq n$, $\frac{d^m}{dt^m} f(\alpha(t))$ consists of derivatives of order $\leq m$ of f and α. Since the derivatives of order $\leq n$ of α and $\overline{\alpha}$ coincide, we obtain

$$\frac{d^m}{dt^m} f(\alpha(t)) = \frac{d^m}{dt^m} f(\overline{\alpha}(t)) = 0$$

at $t = 0$.

Conversely, suppose that $f(\alpha(0)) = 0$ and $df(\alpha(t))/dt = ... = d^n f(\alpha(t))/dt^n = 0$ at $t = 0$. It is possible to assume, by renaming the axes if necessary, that $f_z \neq 0$, then we can represent the surface S as the graph of differentiable function $z = g(x, y)$. Now we define the curve $\overline{C}$ as

$$\overline{x}(t) = x(t), \ \overline{y}(t) = y(t), \ \overline{z}(t) = g(x(t), y(t)), \ \overline{\alpha}(t) = (\overline{x}(t), \overline{y}(t), \overline{z}(t)).$$

By definition, $f(\overline{\alpha}(t)) = 0$, so $\overline{C} \subset S$. We assert that C and $\overline{C}$ have contact of order $\geq n$. It is sufficient to show that $z^{(k)}(0) = \overline{z}^{(k)}(0)$ for each $k \leq n$. Since $df(\alpha(t))/dt = df(\overline{\alpha}(t))/dt = 0$ at $t = 0$,

$$f_x x' + f_y y' + f_z z' = f_x \overline{x}' + f_y \overline{y}' + f_z \overline{z}' = 0 \implies f_z(z' - \overline{z}') = 0 \implies z' = \overline{z}'$$

at $t = 0$. Next, $d^2 f(\alpha(t))/dt^2 = d^2 f(\overline{\alpha}(t))/dt^2 = 0$ implies that

$$\left(f_{xx}x' + f_{xy}y' + f_{xz}z'\right)x' + f_x x'' + \left(f_{yx}x' + f_{yy}y' + f_{yz}z'\right)y' + f_y y''$$
$$+ \left(f_{zx}x' + f_{zy}y' + f_{zz}z'\right)z' + f_z z''$$
$$= \left(f_{xx}\overline{x}' + f_{xy}\overline{y}' + f_{xz}\overline{z}'\right)\overline{x}' + f_x \overline{x}'' + \left(f_{yx}\overline{x}' + f_{yy}\overline{y}' + f_{yz}\overline{z}'\right)y' + f_y \overline{y}''$$
$$+ \left(f_{zx}\overline{x}' + f_{zy}\overline{y}' + f_{zz}\overline{z}'\right)\overline{z}' + f_z \overline{z}'',$$

and we can obtain $f_z(z'' - \overline{z}'') = 0 \implies z'' = \overline{z}''$ at $t = 0$. By continuing this process, we can conclude that $z^{(k)}(0) = \overline{z}^{(k)}(0)$ for all $k \leq n$. Therefore C and S have contact of order $\geq n$.

b. Let $f(x, y, z) = ax + by + cz = 0$ be the representation of a plane P, and let $\alpha(t) = (x(t), y(t), z(t))$ be a parametrization of C in $\mathbf{p}$. Then we have $f(\alpha(t)) = ax(t) + by(t) + cz(t) = 0$. By (a), we have $df(\alpha(t))/dt = d^2 f(\alpha(t))/dt^2 = 0$ at $t = 0$, that is,

$$ax'(0) + by'(0) + cz'(0) = 0, \quad ax''(0) + by''(0) + cz''(0) = 0.$$

Therefore $(a, b, c) \cdot \alpha'(0) = (a, b, c) \cdot \alpha''(0) = 0$, which means that the plane P contains the vectors $\alpha'(0)$ and $\alpha''(0)$. Hence P is the osculating plane of C at $\mathbf{p}$.

c. Let $f(x, y, z) = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 - r^2 = 0$ be the representation of the sphere S. Define $\mathbf{x}_0 = (x_0, y_0, z_0)$ so that $f(\alpha(s)) = |\alpha(s) - \mathbf{x}_0|^2 - r^2$. Since a sphere has contact of order ≥ 3 with a curve C at $\mathbf{p}$, by (a), we have $d^m f(\alpha(s))/ds^m = 0$ for $m \leq 3$.

$$\frac{d}{dt}f(\alpha(s)) = (\alpha - \mathbf{x}_0) \cdot \mathbf{t} = 0$$

implies that $(\alpha(0) - \mathbf{x}_0) \cdot \mathbf{t}(0) = 0$.

$$\frac{d^2}{dt^2}f(\alpha(s)) = \mathbf{t} \cdot \mathbf{t} + (\alpha - \mathbf{x}_0) \cdot k\mathbf{n} = 1 + (\alpha - \mathbf{x}_0) \cdot k\mathbf{n} = 0$$

gives that $(\alpha(0) - \mathbf{x}_0) \cdot \mathbf{n}(0) = -1/k(0)$.

$$\frac{d^3}{dt^3}f(\alpha(s)) = \mathbf{t} \cdot k\mathbf{n} + (\alpha - \mathbf{x}_0) \cdot (k'\mathbf{n} + k(-k\mathbf{t} - \tau\mathbf{b}))$$
$$= (\alpha - \mathbf{x}_0) \cdot (k'\mathbf{n} - k^2\mathbf{t} - k\tau\mathbf{b})$$
$$= -k'/k - 0 - (\alpha - \mathbf{x}_0) \cdot (k\tau\mathbf{b}) = 0$$

implies that $(\alpha(0) - \mathbf{x}_0) \cdot \mathbf{b}(0) = -k'(0)/(k(0)^2\tau(0))$. Therefore we have

$$(\alpha(0) - \mathbf{x}_0) = ((\alpha(0) - \mathbf{x}_0) \cdot \mathbf{t})\mathbf{t} + ((\alpha(0) - \mathbf{x}_0) \cdot \mathbf{n})\mathbf{n} + ((\alpha(0) - \mathbf{x}_0) \cdot \mathbf{b})\mathbf{b}$$
$$= -\frac{1}{k}\mathbf{n} - \frac{k'}{k^2\tau}\mathbf{b}.$$

Therefore the center of the sphere is given by

$$\mathbf{x}_0 = \alpha(0) + \frac{1}{k}\mathbf{n} + \frac{k'}{k^2\tau}\mathbf{b}.$$

11. Let $\mathbf{x}(u, v) = (u, v, u^3 - 3v^2 u)$ be the parametrization of the monkey saddle S. Recall that Dupin indicatrix at $\mathbf{p} = (0, 0, 0)$ is the set of vectors $\mathbf{w} \in T_{\mathbf{p}}(S)$ such that $\mathbf{II}_{\mathbf{p}}(\mathbf{w}) = \pm 1$. However, $\mathbf{x}_{uu} = \mathbf{x}_{uv} = \mathbf{x}_{vv} = (0, 0, 0)$ at $\mathbf{p}$, so we have $\mathbf{II}_{\mathbf{p}}(\mathbf{w}) = 0$ for every vector $\mathbf{w} \in T_{\mathbf{p}}(S)$. Hence there is no point such that satisfies the definition of Dupin indicatrix.

Observe that the curve obtained as the intersection of S with a plane parallel to $T_{\mathbf{p}}(S)$ and close to $\mathbf{p}$ is as follows.
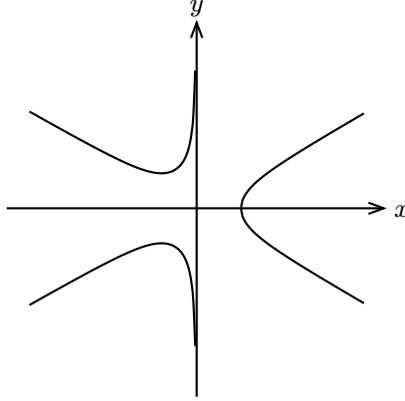
This graph is not similar to quadratic curves. In the argument of Example 5, we can see that the curve C is given by $k_1 x^2 + k_2 y^2 + 2R = 2\varepsilon$, and we consider $k_1 x^2 + k_2 y^2 = 2\varepsilon$ as a first-order approximation of C in a neighborhood of $\mathbf{p}$. This was possible because $\mathbf{p}$ was not a planar point. However, in the case of monkey saddle, $k_1 = k_2 = 0$, so we can't apply this approximation.

12. a. Observe that the curve $v = v_0$ is contained in the plane $x \sin v_0 - y \cos v_0 = 0$, which passes through the z axis. Now we calculate the angle θ between the surface and the plane.

$$\mathbf{x}_u = \left(\cos u \cos v, \cos u \sin v, \frac{\cos^2 u}{\sin u} \right), \quad \mathbf{x}_v = (-\sin u \sin v, \sin u \cos v, \varphi'(v)).$$

Hence $\mathbf{x}_u \wedge \mathbf{x}_v = (\varphi'(v) \cos u \sin v - \cos^2 u \cos v, -\varphi'(v) \cos u \cos v - \cos^2 u \sin v, \cos u \sin u)$ and

$$|\mathbf{x}_u \wedge \mathbf{x}_v| = \sqrt{\varphi'(v)^2 \cos^2 u + \cos^4 u + \cos^2 u \sin^2 u} = \sqrt{\varphi'(v)^2 + 1} |\cos u|.$$

Therefore we obtain

$$\cos \theta = \left| \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|} \cdot (\sin v_0, -\cos v_0, 0) \right|$$

$$= \left| \frac{(\varphi'(v_0) \cos u \sin v_0 - \cos^2 u \cos v_0) \sin v_0 + (-\varphi'(v) \cos u \cos v_0 - \cos^2 u \sin v_0)(-\cos v_0)}{\sqrt{\varphi'(v_0)^2 + 1} |\cos u|} \right|$$

$$= \left| \frac{\varphi'(v_0) \cos u}{\sqrt{\varphi'(v_0)^2 + 1} |\cos u|} \right| = \frac{|\varphi'(v_0)|}{\sqrt{\varphi'(v_0)^2 + 1}},$$

which is constant. Note that the curve $v = v_0$ is a line of curvature of the plane $x \sin v_0 - y \cos v_0 = 0$. Since the angle θ is constant, we conclude that the curves $v = \text{const.}$ are lines of curvature of the surface S by Exercise 15 of Sec. 3-2.

b. The curve $v = v_0$ is generated from a tractrix via a translation parallel to the z-axis followed by a rotation about the z-axis. We already showed the property of the curve that the length of the segment of a tangent line determined by its point of tangency and the z axis is constantly equal to 1, from Exercise 4 of Sec. 1-3.

13. Since S is a regular surface, there is a parametrization $\mathbf{x} : U \to S$ for S at $\mathbf{p}$, which is differentiable and a homeomorphism, and $d\mathbf{x_q}$ is one-to-one for each $\mathbf{q} \in U$. Now define $\overline{\mathbf{x}} : U \to \overline{S}$ as $\overline{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = c\mathbf{x}$. We assert that $\overline{\mathbf{x}}$ is a parametrization for $\overline{S}$ at $\mathbf{F}(\mathbf{p})$. Since $\mathbf{x}$ is differentiable, $c\mathbf{x}$ is also differentiable. $\mathbf{x}$ has an inverse $\mathbf{x}^{-1}$, so $c\mathbf{x}$ also has an inverse $(c\mathbf{x})^{-1} = (1/c)\mathbf{x}^{-1}$. Finally, $d(c\mathbf{x})_\mathbf{q} = c \cdot d\mathbf{x_q}$ is one-to-one for each $\mathbf{q} \in U$. Therefore $\overline{\mathbf{x}}$ is a parametrization for $\overline{S}$ at $\mathbf{F}(\mathbf{p})$, and $\overline{S}$ is a regular surface. By simple calculation,

$$\overline{\mathbf{x}}_u = c\mathbf{x}_u, \ \overline{\mathbf{x}}_v = c\mathbf{x}_v, \ \overline{\mathbf{x}}_{uu} = c\mathbf{x}_{uu}, \ \overline{\mathbf{x}}_{uv} = c\mathbf{x}_{uv}, \ \overline{\mathbf{x}}_{vv} = c\mathbf{x}_{vv}.$$

Therefore

$$\overline{E} = \langle \overline{\mathbf{x}}_u, \overline{\mathbf{x}}_u \rangle = \langle c\mathbf{x}_u, c\mathbf{x}_u \rangle = c^2 \langle \mathbf{x}_u, \mathbf{x}_u \rangle = c^2 E, \quad \overline{F} = c^2 F, \quad \overline{G} = c^2 G.$$

Furthermore,

$$\overline{N} = \frac{\overline{\mathbf{x}}_u \wedge \overline{\mathbf{x}}_v}{|\overline{\mathbf{x}}_u \wedge \overline{\mathbf{x}}_v|} = \frac{c^2 (\mathbf{x}_u \wedge \mathbf{x}_v)}{c^2 |\mathbf{x}_u \wedge \mathbf{x}_v|} = N$$

implies that

$$\overline{e} = \langle \overline{N}, \overline{\mathbf{x}}_{uu} \rangle = c\langle N, \mathbf{x}_{uu} \rangle = ce, \quad \overline{f} = cf, \quad \overline{g} = cg.$$

Therefore we obtain

$$\overline{K} = \frac{\overline{e}\overline{g} - \overline{f}^2}{\overline{E}\overline{G} - \overline{F}^2} = \frac{c^2 (eg - f^2)}{c^4 (EG - F^2)} = \frac{1}{c^2} K, \quad \overline{H} = \frac{1}{2} \frac{\overline{e}\overline{G} - 2\overline{f}\overline{F} + \overline{g}\overline{E}}{\overline{F}\overline{G} - \overline{F}^2} = \frac{1}{2} \frac{c^3 (eG - 2fF + gE)}{c^4 (FG - F^2)} = \frac{1}{c} H.$$

14. Instead of rotating the curve $y = x^3 (-1 < x < 1)$ about the line $x = 1$, we consider the rotation of the curve $y = (x+1)^3 (-2 < x < 0)$ about the line $x = 0$. Let $\mathbf{x} : U \to S$,

$$\mathbf{x}(u, v) = (u \cos v, (u+1)^3, u \sin v), \quad U = \{(u,v) -2 < u < 0, \ 0 < v < 2\pi\}$$

be a parametrization of the given curve. Our goal is to show that the points obtained by rotation of the point $(-1, 0)$ (that is, $u = -1$) are planar points. By computation,

$$\mathbf{x}_u = (\cos v, 3(u+1)^2, \sin v) = (\cos v, 0, \sin v), \ \mathbf{x}_v = (-u \sin v, 0, u \cos v) = (\sin v, 0, -\cos v).$$

Hence $E = 1, \ F = 0, \ G = 1$. Next, we have

$$\mathbf{x}_{uu} = (0, 6(u+1), 0) = (0, 0, 0), \ \mathbf{x}_{uv} = (-\sin v, 0, \cos v),$$

$$\mathbf{x}_{vv} = (-u \cos v, 0, -u \sin v) = (\cos v, 0, \sin v)$$

implies that $e = f = g = 0$. Therefore the points corresponding to $u = -1$ are planar.

15. If there exists such an example, it may locally be written in the form $z = f(x, y)$, with $f(0,0) = 0$, $f_x(0,0) = 0$, $f_y(0,0) = 0$. Then it must be

$$K = \frac{f_{xx} f_{yy} - f_{xy}^2}{\left(1 + f_x^2 + f_y^2\right)^2} = f_{xx} f_{yy} - f_{xy}^2 = 0 \ \text{ iff } \ (x, y) = (0, 0),$$

and

$$H = \frac{(1 + f_x^2) f_{yy} - 2 f_x f_y f_{xy} + (1 + f_y^2) f_{xx}}{\left(1 + f_x^2 + f_y^2\right)^{3/2}} = f_{xx} + f_{yy} \neq 0 \ \text{ when } \ (x, y) = (0, 0).$$

We set $f(x, y) = xy + \alpha(x) + \beta(y)$, where $\alpha(x)$ is a function of x alone and $\beta(y)$ is a function of y alone. Then the condition above is changed into following condition

$$\alpha_{xx} \beta_{yy} = 1 \ \text{ iff } \ (x, y) = (0, 0), \quad \alpha_{xx} + \beta_{yy} \neq 0 \ \text{ when } \ (x, y) = (0, 0).$$

We can see that the functions $\alpha_{xx} = \cos x$ and $\beta_{yy} = \cos y$ satisfies the equation. Therefore we conclude that $f(x, y) = xy - \cos x - \cos y + 2$ is such an example.

16. We denote the open ball centered at $\mathbf{0}$ of radius r by $\mathbb{S}_r^2$. Since S is compact, there exists some $R > 0$ such that $S \subset \mathbb{S}_R^2$. Hence we can define

$$r_0 = \inf\{r > 0 \mid S \subset \overline{\mathbb{S}_r^2}\}.$$

Since S is compact, S meets the boundary of the sphere $\mathbb{S}_{r_0}^2$ at some point $\mathbf{p} \in S$, and $S \subset \overline{\mathbb{S}_{r_0}^2}$ in the neighborhood of $\mathbf{p}$. Now consider the normal section C at $\mathbf{p}$ along an unit vector $\mathbf{v} \in T_\mathbf{p}(S)$. Since C is contained inside a disk of radius r_0, we can apply Exercise 5 of Sec. 1-7, so that the curvature k of C at $\mathbf{p}$ satisfies $k \geq 1/r_0$. Then we obtain $k_n(\mathbf{v}) = k\langle \mathbf{n}, \mathbf{N} \rangle = k \geq 1/r_0$ because the normal vectors $\mathbf{n}$ and $\mathbf{N}$ coincides at $\mathbf{p}$. Remember that the unit vector $\mathbf{v} \in T_\mathbf{p}(S)$ was arbitrary; hence if we let k_1 and k_2 be the principal curvatures of S at $\mathbf{p}$, then $k_1 \geq 1/r_0$ and $k_2 \geq 1/r_0$. Therefore

$$K(\mathbf{p}) = k_1 k_2 \geq \frac{1}{r_0^2} > 0,$$

and we conclude that $\mathbf{p}$ is an elliptic point of S.

17. We define the Gaussian curvature as follows. Given point $\mathbf{p}$ on the regular surface S, there is a parametrization $\mathbf{x} : U \to S$ for S at $\mathbf{p}$. We now define the Gaussian curvature at $\mathbf{p}$ as $K = \det(d\mathbf{N_p})$, where $\mathbf{N} = \mathbf{x}_u \wedge \mathbf{x}_v$ is the unit normal vector. This definition is well-defined for non-orientable surfaces. For example, consider the Möbius strip. When traversing a non-orientable loop, the local unit normal vector $\mathbf{N}$ returns as $-\mathbf{N}$. However, since the differential of the Gauss map satisfies

$$\det(d(-\mathbf{N})_\mathbf{p}) = \det(-d\mathbf{N_p}) = (-1)^2 \det(d\mathbf{N_p}) = \det(d\mathbf{N_p}),$$

the Gaussian curvature K is invariant under a change of orientation. In contrast, the mean curvature cannot be globally defined on a non-orientable surface, because $H = -\frac{1}{2}\operatorname{tr}(d\mathbf{N_p})$ flips its sign whenever the local orientation is reversed.

18. The partial derivatives of $\mathbf{x}$ are

$$\mathbf{x}_u = \left(-\frac{1}{2}v\cos\frac{u}{2}\sin u + \left(2 - v\sin\frac{u}{2}\right)\cos u, -\frac{1}{2}v\cos\frac{u}{2}\cos u - \left(2 - v\sin\frac{u}{2}\right)\sin u, -\frac{1}{2}v\sin\frac{u}{2}\right),$$

$$\mathbf{x}_v = \left(-\sin\frac{u}{2}\sin u, -\sin\frac{u}{2}\cos u, \cos\frac{u}{2}\right).$$

Hence we obtain $E = \left(-\frac{1}{2}v\cos\frac{u}{2}\right)^2 + \left(2 - v\sin\frac{u}{2}\right)^2 + \left(-\frac{1}{2}v\sin\frac{u}{2}\right)^2 = \frac{1}{4}v^2 + \left(2 - v\sin\frac{u}{2}\right)^2$, $F = 0$ and $G = 1$. Moreover,

$$\mathbf{x}_{uu} = \left(\frac{1}{4}v\sin\frac{u}{2}\sin u - v\cos\frac{u}{2}\cos u - \left(2 - v\sin\frac{u}{2}\right)\sin u,\right.$$

$$\frac{1}{4}v\sin\frac{u}{2}\cos u + v\cos\frac{u}{2}\sin u - \left(2 - v\sin\frac{u}{2}\right)\cos u, -\frac{1}{4}v\cos\frac{u}{2}\right),$$

$$\mathbf{x}_{uv} = \left(-\frac{1}{2}\cos\frac{u}{2}\sin u - \sin\frac{u}{2}\cos u, -\frac{1}{2}\cos\frac{u}{2}\cos u + \sin\frac{u}{2}\sin u, -\frac{1}{2}\sin\frac{u}{2}\right), \quad \mathbf{x}_{vv} = (0,0,0).$$

Since $g = 0$, the Gaussian curvature is $K = (eg - f^2)/(EG - F^2) = -f^2/E$. We now calculate f.

$$f = \frac{(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uv})}{\sqrt{EG - F^2}}$$

$$= \frac{1}{\sqrt{E}}\begin{pmatrix} -\frac{1}{2}v\cos\frac{u}{2}\sin u + \left(2 - v\sin\frac{u}{2}\right)\cos u & -\frac{1}{2}v\cos\frac{u}{2}\cos u - \left(2 - v\sin\frac{u}{2}\right)\sin u & -\frac{1}{2}v\sin\frac{u}{2} \\ -\sin\frac{u}{2}\sin u & -\sin\frac{u}{2}\cos u & \cos\frac{u}{2} \\ -\frac{1}{2}\cos\frac{u}{2}\sin u - \sin\frac{u}{2}\cos u & -\frac{1}{2}\cos\frac{u}{2}\cos u + \sin\frac{u}{2}\sin u & -\frac{1}{2}\sin\frac{u}{2} \end{pmatrix}$$

$$= \frac{1}{\sqrt{E}}\begin{pmatrix} 2\cos u & -2\sin u & 0 \\ -\sin\frac{u}{2}\sin u & -\sin\frac{u}{2}\cos u & \cos\frac{u}{2} \\ -\frac{1}{2}\cos\frac{u}{2}\sin u - \sin\frac{u}{2}\cos u & -\frac{1}{2}\cos\frac{u}{2}\cos u + \sin\frac{u}{2}\sin u & -\frac{1}{2}\sin\frac{u}{2} \end{pmatrix}$$

where we subtract v times of third row from the first row, and consequently,

$$f = \frac{1}{\sqrt{E}}\left\{2\cos u\left(\frac{1}{2}\sin^2\frac{u}{2}\cos u - \left(-\frac{1}{2}\cos\frac{u}{2}\cos u + \sin\frac{u}{2}\sin u\right)\cos\frac{u}{2}\right)\right.$$

$$\left. +2\sin u\left(\frac{1}{2}\sin^2\frac{u}{2}\sin u - \left(-\frac{1}{2}\cos\frac{u}{2}\sin u - \sin\frac{u}{2}\cos u\right)\cos\frac{u}{2}\right)\right\}$$

$$= \frac{1}{\sqrt{E}}\left\{2\cos u\left(\frac{1}{2}\cos u - \sin\frac{u}{2}\cos\frac{u}{2}\sin u\right) + 2\sin u\left(\frac{1}{2}\sin u + \sin\frac{u}{2}\cos\frac{u}{2}\cos u\right)\right\}$$

$$= \frac{1}{\sqrt{E}}(\cos^2 u + \sin^2 u) = \frac{1}{\sqrt{E}}.$$

Therefore we have

$$K = -\frac{f^2}{E} = -\frac{1}{E^2} = -\frac{1}{\{v^2/4 + (2 - v\sin(u/2))^2\}^2}.$$
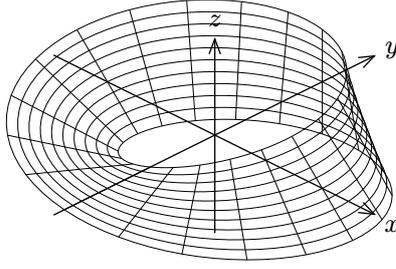
19. We rewrite the equation of the hyperboloid $x^2 + y^2 - z^2 = 1$ as $(x+z)(x-z) = (1-y)(1+y)$. Then for each $k \neq 0$, there are two straight lines

$$
\begin{cases} x + z = k(1+y) \\ x - z = \frac{1}{k}(1-y), \end{cases} \quad \begin{cases} x + z = k(1-y) \\ x - z = \frac{1}{k}(1+y) \end{cases}
$$

which are entirely contained in the surface. Hence these straight lines are the asymptotic curves of the hyperboloid. Now we assert that these are the only asymptotic curves. Since each point $\mathbf{p} \in S$ is a hyperbolic point, there are exactly two asymptotic directions at $\mathbf{p}$, which are uniquely determined by the equation above.

20. We first assert that $\mathbf{p}$ is an umbilical point of S if and only if $d\mathbf{N_p}(\mathbf{v}) \cdot (\mathbf{N} \wedge \mathbf{v}) = 0$ for every $\mathbf{v} \in T_{\mathbf{p}}(S)$. If $\mathbf{p}$ is an umbilical point, then every tangent vector $\mathbf{v}$ is a principal direction at $\mathbf{p}$. Hence we have $d\mathbf{N_p}(\mathbf{v}) = \lambda \mathbf{v}$ and $d\mathbf{N_p}(\mathbf{v}) \cdot (\mathbf{N} \wedge \mathbf{v}) = \lambda \mathbf{v} \cdot (\mathbf{N} \wedge \mathbf{v}) = 0$. Conversely, suppose that $d\mathbf{N_p}(\mathbf{v}) \cdot (\mathbf{N} \wedge \mathbf{v}) = 0$ holds for every $\mathbf{v} \in T_{\mathbf{p}}(S)$. Since $|\mathbf{N}| = 1$, we have $\langle \mathbf{N}', \mathbf{N} \rangle = \langle d\mathbf{N_p}(\mathbf{v}), \mathbf{N} \rangle = 0$. Moreover, $\{\mathbf{N}, \mathbf{v}, \mathbf{N} \wedge \mathbf{v}\}$ is an orthogonal basis of $\mathbb{R}^3$, so it must be $d\mathbf{N_p}(\mathbf{v}) \parallel \mathbf{v}$. Therefore $d\mathbf{N_p}(\mathbf{v}) = \lambda \mathbf{v}$ for every vector $\mathbf{v} \in T_{\mathbf{p}}(S)$ and $\mathbf{p}$ is an umbilical point.

Now consider a point $\mathbf{p} = (x, y, z) \in S$ and tangent vector $\mathbf{v} = (v_1, v_2, v_3) \in T_{\mathbf{p}}(S)$. Since $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ implies $(xx')/a^2 + (yy')/b^2 + (zz')/c^2 = 0$, that is,

$$
\left\langle \left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2} \right), (x', y', z') \right\rangle = 0,
$$

the normal vectors are given by

$$
\mathbf{N}(x, y, z) = \frac{1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}} \left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2} \right) = \frac{1}{f} \left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2} \right) \implies f\mathbf{N} = \left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2} \right).
$$

Then we have

$$
df_{\mathbf{p}}(\mathbf{v})\mathbf{N} + f d\mathbf{N_p}(\mathbf{v}) = \left(\frac{v_1}{a^2}, \frac{v_2}{b^2}, \frac{v_3}{c^2} \right).
$$

The point $\mathbf{p}$ is an umbilical point if and only if $d\mathbf{N}(\mathbf{v}) \cdot (\mathbf{N} \wedge \mathbf{v}) = 0$, which is equivalent to

$$
(df\mathbf{N} + f d\mathbf{N}) \cdot (\mathbf{N} \wedge \mathbf{v}) = 0 \iff \left(\frac{v_1}{a^2}, \frac{v_2}{b^2}, \frac{v_3}{c^2} \right) \cdot (\mathbf{N} \wedge \mathbf{v}) = 0 \iff \frac{1}{f} \det \begin{pmatrix} \frac{v_1}{a^2} & \frac{v_2}{b^2} & \frac{v_3}{c^2} \\ \frac{x}{a^2} & \frac{y}{b^2} & \frac{z}{c^2} \\ v_1 & v_2 & v_3 \end{pmatrix} = 0
$$

$$
\iff -\frac{x}{a^2}\left(\frac{1}{b^2} - \frac{1}{c^2} \right) v_2 v_3 + \frac{y}{b^2}\left(\frac{1}{a^2} - \frac{1}{c^2} \right) v_1 v_3 - \frac{z}{c^2}\left(\frac{1}{a^2} - \frac{1}{c^2} \right) v_1 v_2 = 0.
$$

Note that we can find real number v_3 for every pair of real numbers $(v_1, v_2) \in \mathbb{R}^2$, so that the equation $v_1 x/a^2 + v_2 y/b^2 + v_3 z/c^2 = 0$ holds. Multiply z/c^2 in the both sides of the equation above and apply the equation $v_3 z/c^2 = -(v_1 x/a^2 + v_2 y/b^2)$. Then we obtain

$$
-\frac{xy}{a^2 b^2}\left(\frac{1}{a^2} - \frac{1}{c^2} \right) v_1^2 + \left(\frac{x^2}{a^4}\left(\frac{1}{b^2} - \frac{1}{c^2} \right) - \frac{y^2}{b^4}\left(\frac{1}{a^2} - \frac{1}{c^2} \right) - \frac{z^2}{c^4}\left(\frac{1}{a^2} - \frac{1}{b^2} \right) \right) v_1 v_2 + \frac{xy}{a^2 b^2}\left(\frac{1}{b^2} - \frac{1}{c^2} \right) v_2^2 = 0
$$

for every $v_1, v_2 \in \mathbb{R}$, so it must be the coefficients of the $v_1^2, v_1 v_2, v_2^2$ to be zero.

Consider the case $0 < a < b < c$. Then we have $xy = 0$ from the coefficient of v_1^2. If $x = 0$, then

$$0 > -\frac{y^2}{b^4}\left(\frac{1}{a^2} - \frac{1}{c^2}\right) = \frac{z^2}{c^4}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) > 0,$$

which is a contradiction. So $y = 0$, and we have

$$\frac{x^2}{a^4}\left(\frac{1}{b^2} - \frac{1}{c^2}\right) = \frac{z^2}{c^4}\left(\frac{1}{a^2} - \frac{1}{b^2}\right).$$

Solving this equation with $x^2/a^2 + z^2/c^2 = 1$, we obtain four umbilical points

$$x = \pm a\sqrt{\frac{b^2 - a^2}{c^2 - a^2}}, \quad y = 0, \quad z = \pm c\sqrt{\frac{c^2 - b^2}{c^2 - a^2}}.$$

Next, consider the case $0 < a < b = c$. Similar to the case above, we have $xy = 0$ from the coefficient of v_1^2, and $x = 0$ implies the contradiction. Thus we obtain $y = 0$ and $z = 0$, and there are two umbilical points $(\pm a, 0, 0)$. Similarly, if $0 < a = b < c$, there are two umbilical points $(0, 0, \pm c)$.

Finally, if $0 < a = b = c$, then the surface becomes a sphere, and every point is an umbilical point.

21. **a.** Given $\mathbf{p} \in S$ and $\mathbf{w} \in T_{\mathbf{p}}(S)$, let $\alpha : I \to S$ be a curve such that $\alpha(0) = \mathbf{p}$ and $\alpha'(0) = \mathbf{w}$, then

$$d(f\mathbf{N})_{\mathbf{p}}(\mathbf{w}) = \frac{d}{dt}(f\mathbf{N} \circ \alpha)(t)\Big|_{t=0} = \frac{d}{dt}(f \circ \alpha)(t) \cdot (\mathbf{N} \circ \alpha)(t)\Big|_{t=0}$$

$$= (f \circ \alpha)'(0)(\mathbf{N} \circ \alpha)(0) + (f \circ \alpha)(0)(\mathbf{N} \circ \alpha)'(0)$$

$$= df_{\mathbf{p}}(\mathbf{w})\mathbf{N}(\mathbf{p}) + f(\mathbf{p})d\mathbf{N}_{\mathbf{p}}(\mathbf{w}).$$

Therefore we have

$$d(f\mathbf{N})(\mathbf{v}_1) \wedge d(f\mathbf{N})(\mathbf{v}_2) = (df(\mathbf{v}_1)\mathbf{N} + fd\mathbf{N}(\mathbf{v}_1)) \wedge (df(\mathbf{v}_2)\mathbf{N} + fd\mathbf{N}(\mathbf{v}_2))$$

$$= fdf(\mathbf{v}_1)\mathbf{N} \wedge d\mathbf{N}(\mathbf{v}_2) + fdf(\mathbf{v}_2)d\mathbf{N}(\mathbf{v}_1) \wedge \mathbf{N} + f^2 d\mathbf{N}(\mathbf{v}_1) \wedge d\mathbf{N}(\mathbf{v}_2)$$

and

$$\langle d(f\mathbf{N})(\mathbf{v}_1) \wedge d(f\mathbf{N})(\mathbf{v}_2), f\mathbf{N}\rangle = \langle f^2 d\mathbf{N}(\mathbf{v}_1) \wedge d\mathbf{N}(\mathbf{v}_2), f\mathbf{N}\rangle = f^3\langle d\mathbf{N}(\mathbf{v}_1) \wedge d\mathbf{N}(\mathbf{v}_2), \mathbf{N}\rangle$$

$$= f^3\langle \det(d\mathbf{N})(\mathbf{v}_1 \wedge \mathbf{v}_2), \mathbf{N}\rangle = f^3 \det(d\mathbf{N}) = f^3 K.$$

Hence the Gaussian curvature K is given by $K = \langle d(f\mathbf{N})(\mathbf{v}_1) \wedge d(f\mathbf{N})(\mathbf{v}_2), f\mathbf{N}\rangle/f^3$.

b. Since $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ implies $(xx')/a^2 + (yy')/b^2 + (zz')/c^2 = 0$, that is,

$$\left\langle \left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right), (x', y', z') \right\rangle = 0,$$

the normal vectors are given by

$$\mathbf{N}(x, y, z) = \frac{1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}\left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right) = \frac{1}{f}\left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right) \implies f\mathbf{N} = \left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right).$$

Let $\mathbf{v}_i = (\alpha_i, \beta_i, \gamma_i)$ for $i = 1, 2$, Then

$$\langle d(f\mathbf{N})(\mathbf{v}_1) \wedge d(f\mathbf{N})(\mathbf{v}_2), f\mathbf{N}\rangle = \left|\begin{pmatrix} d(f\mathbf{N})(\mathbf{v}_1) \\ d(f\mathbf{N})(\mathbf{v}_2) \\ f\mathbf{N} \end{pmatrix}\right| = \left|\begin{pmatrix} \frac{\alpha_1}{a^2} & \frac{\beta_1}{b^2} & \frac{\gamma_1}{c^2} \\ \frac{\alpha_2}{a^2} & \frac{\beta_2}{b^2} & \frac{\gamma_2}{c^2} \\ \frac{x}{a^2} & \frac{y}{b^2} & \frac{z}{c^2} \end{pmatrix}\right| = \frac{1}{a^2b^2c^2}\left|\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ x & y & z \end{pmatrix}\right|$$

$$= \frac{1}{a^2b^2c^2}\langle \mathbf{v}_1 \wedge \mathbf{v}_2, (x, y, z)\rangle = \frac{1}{a^2b^2c^2}\langle \mathbf{N}, (x, y, z)\rangle$$

$$= \frac{1}{a^2b^2c^2}\left\langle \frac{1}{f}\left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right), (x, y, z)\right\rangle = \frac{1}{a^2b^2c^2 f}.$$

Therefore we have

$$K = \frac{1}{f^3}\langle d(f\mathbf{N})(\mathbf{v}_1) \wedge d(f\mathbf{N})(\mathbf{v}_2), f\mathbf{N}\rangle = \frac{1}{a^2b^2c^2 f^4}.$$

22. **a.** Let $\alpha(t) = \mathbf{x}(u(t), v(t))$, so that $\alpha'(0) = \mathbf{x}_u u' + \mathbf{x}_v v' = \mathbf{w}$. Since $\mathbf{p}$ is a critical point of $\mathbf{h}$, we have $h_u(\mathbf{p}) = h_v(\mathbf{p}) = 0$. Therefore we obtain

$$H_{\mathbf{p}}h(\mathbf{w}) = \frac{d^2}{dt^2}(h \circ \alpha)(t)\Big|_{t=0} = \frac{d}{dt}(h_u u' + h_v v')\Big|_{t=0}$$

$$= (h_{uu}u' + h_{uv}v')u' + h_u u'' + (h_{vu}u' + h_{vv}v')v' + h_v v''$$

$$= (h_{uu}u' + h_{uv}v')u' + (h_{vu}u' + h_{vv}v')v'$$

$$= h_{uu}(\mathbf{p})(u')^2 + 2h_{uv}(\mathbf{p})u'v' + h_{vv}(\mathbf{p})(v')^2.$$

We assert that $H_{\mathbf{p}}h$ does not depend on the choice of $\mathbf{x}$. Let $\overline{\mathbf{x}}(\overline{u}, \overline{v})$ be a another parametrization of S at $\mathbf{p}$, then

$$H_{\mathbf{p}}h(\overline{\mathbf{x}}_{\overline{u}}\overline{u}' + \overline{\mathbf{x}}_{\overline{v}}\overline{v}') = h_{\overline{u}\overline{u}}(\mathbf{p})(\overline{u}')^2 + 2h_{\overline{u}\overline{v}}(\mathbf{p})\overline{u}'\overline{v}' + h_{\overline{v}\overline{v}}(\mathbf{p})(\overline{v}')^2$$

$$= (\overline{u}'\ \overline{v}')\begin{pmatrix} h_{\overline{u}\overline{u}} & h_{\overline{u}\overline{v}} \\ h_{\overline{u}\overline{v}} & h_{\overline{v}\overline{v}} \end{pmatrix}\begin{pmatrix} \overline{u}' \\ \overline{v}' \end{pmatrix}$$

$$= (u'\ v')\begin{pmatrix} \frac{\partial \overline{u}}{\partial u} & \frac{\partial \overline{v}}{\partial u} \\ \frac{\partial \overline{u}}{\partial v} & \frac{\partial \overline{v}}{\partial v} \end{pmatrix}\begin{pmatrix} h_{\overline{u}\overline{u}} & h_{\overline{u}\overline{v}} \\ h_{\overline{u}\overline{v}} & h_{\overline{v}\overline{v}} \end{pmatrix}\begin{pmatrix} \frac{\partial \overline{u}}{\partial u} & \frac{\partial \overline{u}}{\partial v} \\ \frac{\partial \overline{v}}{\partial u} & \frac{\partial \overline{v}}{\partial v} \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix}$$

$$= (u'\ v')\begin{pmatrix} h_{uu} & h_{uv} \\ h_{uv} & h_{vv} \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix}$$

$$= h_{uu}(\mathbf{p})(u')^2 + 2h_{uv}(\mathbf{p})u'v' + h_{vv}(\mathbf{p})(v')^2.$$

Hence $H_{\mathbf{p}}h : T_{\mathbf{p}}(S) \to \mathbb{R}$ is a well-defined quadratic form on $T_{\mathbf{p}}(S)$.

b. We first show that $\mathbf{p}$ is a critical point of the height function $h(\mathbf{q}) = \langle \mathbf{q} - \mathbf{p}, \mathbf{N}(\mathbf{p}) \rangle$. Given $\mathbf{w} \in T_{\mathbf{p}}(S)$, let $\alpha : I \to S$ be a parametrization of a curve in S such that $\alpha(0) = \mathbf{p}$ and $\alpha'(0) = \mathbf{w}$. Then

$$dh_{\mathbf{p}}(\mathbf{w}) = \frac{d}{dt}(h \circ \alpha)(t)\Big|_{t=0} = \frac{d}{dt}\langle \alpha(t) - \mathbf{p}, \mathbf{N}(\mathbf{p}) \rangle\Big|_{t=0} = \langle \alpha'(t), \mathbf{N}(\mathbf{p}) \rangle|_{t=0} = \langle \mathbf{w}, \mathbf{N}(\mathbf{p}) \rangle.$$

Since $\mathbf{w} \in T_{\mathbf{p}}(S)$ is perpendicular to $\mathbf{N}(\mathbf{p})$, we have $dh_{\mathbf{p}}(\mathbf{w}) = 0$ for every $\mathbf{w}$, and that $\mathbf{p}$ is a critical point of h. Now if $|\mathbf{w}| = 1$, then we can suppose that α is parametrized by an arc length, therefore

$$H_{\mathbf{p}}h(\mathbf{w}) = \frac{d^2}{ds^2}(h \circ \alpha)(t)\Big|_{s=0} = \frac{d^2}{ds^2}\langle \alpha(s) - \mathbf{p}, \mathbf{N}(\mathbf{p}) \rangle\Big|_{s=0} = \frac{d}{ds}\langle \alpha'(s), \mathbf{N}(\mathbf{p}) \rangle\Big|_{s=0}$$

$$= \langle \alpha''(s), \mathbf{N}(\mathbf{p}) \rangle|_{s=0} = \langle \alpha''(0), \mathbf{N}(\mathbf{p}) \rangle = \langle k\mathbf{n}(\mathbf{p}), \mathbf{N}(\mathbf{p}) \rangle = k_n(\mathbf{w}) = \mathbf{II}_{\mathbf{p}}(\mathbf{w}).$$

Hence we conclude that Hessian at $\mathbf{p}$ of the height function relative to $T_{\mathbf{p}}(S)$ is the second fundamental form of S at $\mathbf{p}$.

23. a. Given tangent vector $\mathbf{w} \in T_{\mathbf{p}}(S)$, there exists a parametrized curve $\alpha : (-\varepsilon, \varepsilon) \to S$ such that $\alpha(0) = \mathbf{p}$ and $\alpha'(0) = \mathbf{w}$. Then

$$d(h_{\mathbf{r}})_{\mathbf{p}}(\mathbf{w}) = \frac{d}{dt}(h_{\mathbf{r}} \circ \alpha)(t)\Big|_{t=0} = \frac{d}{dt}\langle \alpha(t) - \mathbf{r}, \alpha(t) - \mathbf{r} \rangle\Big|_{t=0}$$

$$= \frac{2\langle \alpha'(t), \alpha(t) - \mathbf{r} \rangle}{2\sqrt{\langle \alpha(t) - \mathbf{r}, \alpha(t) - \mathbf{r} \rangle}}\Big|_{t=0} = \frac{\langle \alpha'(0), \alpha(0) - \mathbf{r} \rangle}{\sqrt{\langle \alpha(0) - \mathbf{r}, \alpha(0) - \mathbf{r} \rangle}} = \frac{\langle \mathbf{w}, \mathbf{p} - \mathbf{r} \rangle}{h_{\mathbf{r}}(\mathbf{p})}.$$

This differential is identically zero if and only if $\langle \mathbf{w}, \mathbf{p} - \mathbf{r} \rangle = 0$ for every $\mathbf{w} \in T_{\mathbf{p}}(S)$, that is, the straight line $\mathbf{pr}$ is normal to S at $\mathbf{p}$.

b. By the definition of Hessian,

$$H_{\mathbf{p}}h_{\mathbf{r}}(\mathbf{w}) = \frac{d^2}{ds^2}(h_{\mathbf{r}} \circ \alpha)(s)\Big|_{s=0} = \frac{d}{ds}\frac{\langle \alpha'(s), \alpha(s) - \mathbf{r} \rangle}{h_{\mathbf{r}}(\alpha(s))}\Big|_{s=0}$$

$$= \frac{(\langle \alpha''(s), \alpha(s) - r \rangle + \langle \alpha'(s), \alpha'(s) \rangle h_{\mathbf{r}}(\alpha(s))) - \langle \alpha'(s), \alpha(s) - \mathbf{r} \rangle h_{\mathbf{r}}'(\alpha(s))}{(h_{\mathbf{r}}(\alpha(s)))^2}\Big|_{s=0}.$$

Since $\mathbf{p}$ is a critical point of $h_{\mathbf{r}}$, we have $\langle \alpha'(0), \alpha(0) - \mathbf{r} \rangle = 0$. Therefore

$$H_{\mathbf{p}}h_{\mathbf{r}}(\mathbf{w}) = \frac{1 + \langle \alpha''(0), \mathbf{p} - \mathbf{r} \rangle}{h_{\mathbf{r}}(\mathbf{p})} = \frac{1 + \langle \alpha''(0), -h_{\mathbf{r}}(\mathbf{p})\mathbf{N} \rangle}{h_{\mathbf{r}}(\mathbf{p})}$$

$$= \frac{1}{h_{\mathbf{r}}(\mathbf{p})} - \langle \alpha''(0), \mathbf{N} \rangle = \frac{1}{h_{\mathbf{r}}(\mathbf{p})} - k_n(\mathbf{w}),$$

where we defined the orientation of S by $\mathbf{p} - \mathbf{r} = -h_{\mathbf{r}}(\mathbf{p})\mathbf{N}$. Now observe that

$$\langle A_{\mathbf{p}}h_{\mathbf{r}}(\mathbf{w}), \mathbf{w} \rangle = H_{\mathbf{p}}h_{\mathbf{r}}(\mathbf{w}) = \frac{1}{h_{\mathbf{r}}(\mathbf{p})}\langle \mathbf{w}, \mathbf{w} \rangle + \langle d\mathbf{N}_{\mathbf{p}}(\mathbf{w}), \mathbf{w} \rangle = \left\langle \left(\frac{1}{h_{\mathbf{r}}(\mathbf{p})}I + d\mathbf{N}_{\mathbf{p}} \right)(\mathbf{w}), \mathbf{w} \right\rangle.$$

Since $A_{\mathbf{p}}h_{\mathbf{r}}$ and $\frac{1}{h_{\mathbf{r}}(\mathbf{p})}I + d\mathbf{N}_{\mathbf{p}}$ are both self-adjoint linear map, it follows that they are the same. Furthermore, for the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, we have

$$A_{\mathbf{p}}h_{\mathbf{r}}(\mathbf{e}_i) = \left(\frac{1}{h_{\mathbf{r}}(\mathbf{p})}I + d\mathbf{N}_{\mathbf{p}} \right)(\mathbf{e}_i) = \left(\frac{1}{h_{\mathbf{r}}(\mathbf{p})} - k_i \right)\mathbf{e}_i, \quad i = 1, 2.$$

Therefore $\{\mathbf{e}_1, \mathbf{e}_2\}$ diagonalizes $A_{\mathbf{p}}h_{\mathbf{r}}$. Note that $\mathbf{p}$ is a degenerate critical point of $h_{\mathbf{r}}$ if and only if one of the eigen values are zero, that is, $h_{\mathbf{r}}(\mathbf{p}) = 1/k_i$ for some $i = 1, 2$.

c. Given point $\mathbf{p} \in S$, there exists a parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ for S at $\mathbf{p}$. Then we define a function $\mathbf{F^p} : U \times \mathbb{R} \to \mathbb{R}^3$ as

$$\mathbf{F^p}(u, v, t) = \mathbf{x}(u, v) + t\mathbf{N}(u, v),$$

where $\mathbf{N}(u, v) = (\mathbf{x}_u \wedge \mathbf{x}_v)/|\mathbf{x}_u \wedge \mathbf{x}_v|$. We first assert that if $h_{\mathbf{r}}$ has degenerate critical point $\mathbf{p}$, then $\mathbf{r}$ is a critical value of $\mathbf{F^p}$. Suppose that $\mathbf{p}$ is a degenerate critical point of $h_{\mathbf{r}}$. Since $\mathbf{p}$ is a critical point, we have $\mathbf{r} - \mathbf{p} = t_0\mathbf{N}(\mathbf{p})$ for some t_0. Since it is degenerate, by part (b), we have $t_0 = 1/k_i$ for some $i = 1, 2$. Let $\mathbf{p} = \mathbf{x}(u_0, v_0)$ and $\mathbf{q} = (u_0, v_0, t_0)$. Now observe that

$$(\mathbf{F^p})_u = \mathbf{x}_u + t\mathbf{N}_u = (1 + ta_{11})\mathbf{x}_u + ta_{21}\mathbf{x}_v,$$

$$(\mathbf{F^p})_v = \mathbf{x}_v + t\mathbf{N}_v = ta_{12}\mathbf{x}_u + (1 + ta_{22})\mathbf{x}_v,$$

$$(\mathbf{F^p})_t = N.$$

Then

$$\det\left(d(\mathbf{F^p})_{\mathbf{q}} \right) = \left((\mathbf{F^p})_u \wedge (\mathbf{F^p})_v \right) \cdot (\mathbf{F^p})_t$$

$$= ((1 + t_0a_{11})(1 + t_0a_{22}) - t_0^2a_{12}a_{21})(\mathbf{x}_u \wedge \mathbf{x}_v) \cdot \mathbf{N}$$

$$= (1 - k_1t_0)(1 - k_2t_0)|\mathbf{x}_u \wedge \mathbf{x}_v| = 0,$$

thus $\mathbf{q}$ is critical point of $\mathbf{F^p}$, and $\mathbf{F^p}(\mathbf{q}) = \mathbf{x}(u_0, v_0) + t_0\mathbf{N}(u_0, v_0) = \mathbf{p} + t_0\mathbf{N}(\mathbf{p}) = \mathbf{r}$ is a critical value of $\mathbf{F^p}$. Hence we showed that if $h_{\mathbf{r}}$ has degenerate critical point $\mathbf{p}$, then $\mathbf{r}$ is a critical value of $\mathbf{F^p}$. If we consider the contrapositive of the proposition above, we know that if $\mathbf{r}$ is not a critical value of $\mathbf{F^p}$, then $\mathbf{p}$ is not a critical point of $h_{\mathbf{r}}$ or $\mathbf{p}$ is not a degenerate critical point of $\mathbf{p}$.

By Sard's theorem, the set $\mathrm{CV}(\mathbf{F^p})$ consists of critical values of $\mathbf{F^p}$ has Lebesgue measure 0 in $\mathbb{R}^3$. Since S is a regular surface, we can cover the surface S by countably many numbers of neighborhoods $\{V_i\}$ of points $\{\mathbf{p}_i\} \subset S$. Then its countable union $\bigcup_i \mathrm{CV}(\mathbf{F^{p_i}})$ also has Lebesgue measure 0 in $\mathbb{R}^3$, so it has empty interior. Now if $\mathbf{r} \in \mathbb{R}^3$ is not contained in following set

$$S \cup \left(\bigcup_i \mathrm{CV}(\mathbf{F^{p_i}}) \right),$$

then we can guarantee that $h_{\mathbf{r}}$ is a Morse function. Since this set has empty interior, we conclude that $B = \{\mathbf{r} \in \mathbb{R}^3 \mid h_{\mathbf{r}} \text{ is a Morse function}\}$ is a dense set in $\mathbb{R}^3$.

24. a. Let $\mathbf{x}(u, v)$ be a parametrization at $\mathbf{p}$, with $\mathbf{x}(0, 0) = \mathbf{p}$. The distance d from a point $q = \mathbf{x}(u, v)$ to the tangent plane $T_{\mathbf{p}}(S)$ is given by

$$d = \langle \mathbf{x}(u, v) - \mathbf{x}(0, 0), \mathbf{N}(\mathbf{p}) \rangle = \frac{1}{2}\mathbf{II}_{\mathbf{p}}(\mathbf{w}) + \mathbf{R},$$

where $\mathbf{w} = \mathbf{x}_u u + \mathbf{x}_v v$, $\lim_{\mathbf{w} \to 0} \mathbf{R}/|\mathbf{w}|^2 \to 0$. (See proof of Prop. 1.) Then we have

$$\frac{d}{|\mathbf{w}|^2} = \frac{1}{2}\frac{\mathbf{II_p(w)}}{|\mathbf{w}|^2} + \frac{\mathbf{R}}{|\mathbf{w}|^2} = \frac{1}{2}\mathbf{II_p}\left(\frac{\mathbf{w}}{|\mathbf{w}|}\right) + \frac{\mathbf{R}}{|\mathbf{w}|^2} = \frac{1}{2}k_n\left(\frac{\mathbf{w}}{|\mathbf{w}|}\right) + \frac{\mathbf{R}}{|\mathbf{w}|^2}.$$

Since principal curvatures k_1 and k_2 have the same sign, normal curvatures $k_n(\mathbf{w}) = k_1\cos^2\theta + k_2\sin^2\theta$ has a fixed sign. We can take neighborhood V of $(0,0)$ sufficiently small, so that if $(u,v) \in V$ then $\mathbf{w}$ is small enough to satisfy

$$\inf_{|\mathbf{v}|=1}\left|\frac{1}{2}k_n(\mathbf{v})\right| > \left|\frac{\mathbf{R}}{|\mathbf{w}|^2}\right|.$$

Then d has same sign as $k_n(\mathbf{w})$, that is, all such (u,v) belong to the same side of $T_\mathbf{p}(S)$.

b. We prove the contrapositive, that is, if the principal curvatures at $\mathbf{p}$ have different signs, then S is not locally convex at $\mathbf{p}$. Similar to part (a), we can write

$$\frac{d}{|\mathbf{w}|^2} = \frac{1}{2}k_n\left(\frac{\mathbf{w}}{|\mathbf{w}|}\right) + \frac{\mathbf{R}}{|\mathbf{w}|^2},$$

and d has same sign as $k_n(\mathbf{w})$ for all (u,v) sufficiently near $(0,0)$. Note that there exist orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2\}$ such that $k_n(\mathbf{e}_1) = k_1$ and $k_n(\mathbf{e}_2) = k_2$ have different signs. For each neighborhood V of $(0,0)$, we can find small $t_1, t_2 \in \mathbb{R}$ so that

$$\mathbf{w} = t_1\mathbf{e}_1, \quad \overline{\mathbf{w}} = t_2\mathbf{e}_2 \in V.$$

Then we can see that

$$\frac{1}{2}k_n\left(\frac{\mathbf{w}}{|\mathbf{w}|}\right) = \frac{1}{2}k_n(\mathbf{e}_1) = \frac{1}{2}k_1, \quad \frac{1}{2}k_n\left(\frac{\overline{\mathbf{w}}}{|\overline{\mathbf{w}}|}\right) = \frac{1}{2}k_n(\mathbf{e}_2) = \frac{1}{2}k_2$$

have different signs, that is, such points belong to distinct sides of $T_\mathbf{p}(S)$. Therefore S is not locally convex at $\mathbf{p}$.

c. Let $\mathbf{x}: U \subset \mathbb{R}^2 \to S$, $\mathbf{x}(u,v) = (u,v,u^3(1+v^2))$ be a parametrization for the surface, where $U = \{(u,v) \mid v^2 < 1/2\}$. Since

$$f_{uu} = 6u(1+v^2), \quad f_{uv} = 6u^2v, \quad f_{vv} = 2u^3,$$

we have

$$K(u,v) = \frac{f_{uu}f_{vv} - f_{uv}^2}{(1+f_u^2+f_v^2)^2} = \frac{12u^4(1-2v^2)}{(1+f_u^2+f_v^2)^2} \geq 0$$

for every $(u,v) \in U$. However, since $\mathbf{x}(u,0) = (u,0,u^3)$ does not have a fixed sign in the neighborhood of $(0,0)$, S is not locally convex at $\mathbf{p} = (0,0,0)$.

d. Choose a coordinate system in $\mathbb{R}^3$ so that the origin O is at $\mathbf{p} \in S$, the xy plane agrees with $T_\mathbf{p}(S)$, and the positive direction of the z axis agrees with the orientation of S at $\mathbf{p}$. Furthermore, choose the x and y axis in $T_\mathbf{p}(S)$ along the principal directions at $\mathbf{p}$. If V is sufficiently small, it can be represented as the graph of a differentiable function

$$z = f(x,y), \quad (x,y) \in D \subset \mathbb{R}^2,$$

where D is an open disk in $\mathbb{R}^2$ and

$$f_x(0,0) = f_y(0,0) = f_{xy}(0,0) = 0, \quad f_{xx}(0,0) = k_1, \quad f_{yy}(0,0) = k_2.$$

We can assume, without loss of generality, that $k_1 \geq 0$ and $k_2 \geq 0$ on D, and we want to prove that $f(x,y) \geq 0$ on D. Assume that $f(\overline{x},\overline{y}) < 0$ for some $(\overline{x},\overline{y}) \in D$. Consider the function

$$h_0(t) = f(t\overline{x}, t\overline{y}), \quad 0 \leq t \leq 1.$$

Since $h_0(0) = 0$ and $h_0(1) < 0$, there exists $t_1 \in (0,1)$ such that $h_0'(t_1) < 0$. Moreover, $h_0'(0) = 0$, so there exists $t_2 \in (0,1)$ such that $h_0''(t_2) < 0$. Let $\mathbf{p}_2 = (t_2\overline{x}, t_2\overline{y}, f(t_2\overline{x}, t_2\overline{y})) \in S$, and consider the height function h_2 of V relative to the tangent plane $T_{\mathbf{p}_2}(S)$ at $\mathbf{p}_2$. Restricted to the curve $\alpha(t) = (t\overline{x}, t\overline{y}, f(t\overline{x}, t\overline{y}))$, this height function is $h_2(t) = \langle \alpha(t) - \mathbf{p}_2, \mathbf{N}_2 \rangle$, where $\mathbf{N}_2$ is the unit normal vector at $\mathbf{p}_2$. Thus, $h_2''(t) = \langle \alpha''(t), \mathbf{N} \rangle$, and at $t = t_2$,

$$h_2''(t_2) = \big\langle (0, 0, h_0''(t_2)), (-f_x(\mathbf{p}_2), -f_y(\mathbf{p}_2), 1) \big\rangle = h_0''(t_2) < 0.$$

But $h_2''(t_2) = \langle \alpha''(t_2), \mathbf{N}_2 \rangle$ is, up to a positive factor, the normal curvature at $\mathbf{p}_2$, in the direction of $\alpha'(t_2)$. This is a contradiction.

4-2. Isometries; Conformal Maps

1. a. First observe that $\mathbf{F}(U)$ is contained in a cone C with the vertex at the origin and $2\alpha \in (0, \pi)$ as the angle of the vertex, because

$$k\sqrt{x^2 + y^2} = ku \sin \alpha = u \cos \alpha = z,$$

where $k = \cot \alpha$. Now we show that for every point $\mathbf{p} \in C$, there exists a neighborhood V of $\mathbf{p}$ in C such that $\mathbf{x} = \mathbf{F}|_V$ is a parametrization at $\mathbf{p}$ for C. Choose arbitrary point $\mathbf{p} \in C$, and take V small enough so that the function $\pi_2(V) \to \mathbb{S}^1$, $v \mapsto (\cos v, \sin v)$ can be one-to-one. (i) $\mathbf{x} = \mathbf{F}|_V$ is differentiable because it is restriction of a differentiable function. (iii) Since

$$d\mathbf{F_q} = \begin{pmatrix} \sin \alpha \cos v & -u \sin \alpha \sin v \\ \sin \alpha \sin v & u \sin \alpha \cos v \\ \cos \alpha & 0 \end{pmatrix}, \quad \frac{\partial(x, y)}{\partial(u, v)} = u \sin^2 \alpha > 0,$$

the differential $d\mathbf{x_q}$ is one-to-one for each $\mathbf{q} \in V$. (ii) Let

$$\mathbf{F}(u, v) = (u \sin \alpha \cos v, u \sin \alpha \sin v, u \cos \alpha) = (x, y, z).$$

Since $u = z/\cos \alpha$ and $(\cos v, \sin v) = \left(\frac{x}{u \sin \alpha}, \frac{y}{u \sin \alpha} \right)$ determines $v \in \mathbb{R}$, $\mathbf{x}$ is one-to-one. By Prop. 4 of Sec 2-2, $\mathbf{x}^{-1}$ is continuous. Therefore $\mathbf{x}$ is a parametrization. Furthermore, $\mathbf{x}^{-1}$ is differentiable; for arbitrary parametrization $\mathbf{y} : W \in \mathbb{R}^2 \to C$, the change of the parametrization

$$\mathbf{x}^{-1} \circ \mathbf{y} : \mathbf{y}^{-1}(N) \to \mathbf{x}^{-1}(N),$$

where $N = \mathbf{x}(V) \cap \mathbf{y}(W)$, is differentiable. Thus we proved that $\mathbf{x} = \mathbf{F}|_V$ is a diffeomorphism. We conclude that $\mathbf{F}$ is a local diffeomorphism of U onto a cone C.

b. $\mathbf{F}_u = (\sin \alpha \cos v, \sin \alpha \sin v, \cos \alpha)$ and $\mathbf{F}_v = (-u \sin \alpha \sin v, u \sin \alpha \cos v)$ implies that

$$E = 1, \quad F = 0, \quad G = u^2 \sin^2 \alpha.$$

Since $G \neq 1$, $\mathbf{F}$ is not a local isometry.

2. We first show that $\overline{\mathbf{x}} = \varphi \circ \mathbf{x}$ is a parametrization for $\overline{S}$ at $\varphi(\mathbf{p})$. (i) $\overline{\mathbf{x}} = \varphi \circ \mathbf{x}$ is differentiable, since φ is differentiable map on S. (ii) $\overline{\mathbf{x}}^{-1} = (\varphi \circ \mathbf{x})^{-1} = \mathbf{x}^{-1} \circ \varphi^{-1}$ is continuous because $\mathbf{x}$ is homeomorphism and φ is diffeomorphism. (iii) We assert that $d\overline{\mathbf{x}}_{\mathbf{q}} = d(\varphi \circ \mathbf{x})_{\mathbf{q}} = d\varphi_{\mathbf{x(q)}} \circ d\mathbf{x_q}$ is one-to-one. $d\mathbf{x_q}$ is one-to-one since $\mathbf{x}$ is parametrization for S. To show that $d\varphi_{\mathbf{x(q)}}$ is one-to-one, observe that $\varphi^{-1} \circ \varphi = \mathrm{Id}_S$ implies that

$$d(\varphi^{-1})_{\varphi(\mathbf{p})} \circ d\varphi_{\mathbf{p}} = d(\mathrm{Id}_S)_{\mathbf{p}} = \mathrm{Id}_{T_{\mathbf{p}}(S)}.$$

Since $\mathrm{Id}_{T_{\mathbf{p}}(S)}$ is one-to-one, $d\varphi_{\mathbf{p}}$ is also one-to-one for each $\mathbf{p} \in S$. Therefore $d\overline{\mathbf{x}}_{\mathbf{q}}$ is one-to-one, which is a composition of one-to-one mappings. Thus $\overline{\mathbf{x}}$ is a parametrization for $\overline{S}$ at $\varphi(\mathbf{p})$.

Since φ is an isometry,

$$\overline{E} = \langle \overline{\mathbf{x}}_u, \overline{\mathbf{x}}_u \rangle = \langle (\varphi \circ \mathbf{x})_u, (\varphi \circ \mathbf{x})_u \rangle = \langle d\varphi(\mathbf{x}_u), d\varphi(\mathbf{x}_u) \rangle = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = E.$$

Similarly, we obtain $\overline{F} = F$, $\overline{G} = G$.

3. First suppose that $\varphi : S \to \overline{S}$ is an isometry. For any parametrized curve $\alpha : [a, b] \subset I \to S$, let $\overline{\alpha} = \varphi \circ \alpha : I \to \overline{S}$ be a parametrized curve in $\overline{S}$, then

$$l(\alpha) = \int_a^b \sqrt{\mathbf{I}_{\alpha(t)}(\alpha'(t))}\, dt = \int_a^b \sqrt{\mathbf{I}_{(\varphi \circ \alpha)(t)}\big(d\varphi_{\alpha(t)}(\alpha'(t)) \big)}\, dt = \int_a^b \sqrt{\mathbf{I}_{\overline{\alpha}(t)}(\overline{\alpha}'(t))}\, dt = l(\overline{\alpha}).$$

Hence the arc length of any parametrized curve α is equal to the arc length of the image curve $\varphi \circ \alpha$.

Conversely, suppose the conclusion of the previous part. For any $\mathbf{p} \in S$ and $\mathbf{0} \neq \mathbf{v} \in T_{\mathbf{p}}(S)$, consider a curve $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha'(0) = \mathbf{v}$. We assert that

$$\mathbf{I}_{(\varphi \circ \alpha)(0)}\big(d\varphi_{\alpha(0)}(\alpha'(0))\big) = \mathbf{I}_{\alpha(0)}(\alpha'(0)).$$

Otherwise, if $\mathbf{I}_{(\varphi \circ \alpha)(0)}\big(d\varphi_{\alpha(0)}(\alpha'(0))\big) > \mathbf{I}_{\alpha(0)}(\alpha'(t))$, then there exists a neighborhood J of 0 in $(-\varepsilon, \varepsilon)$ such that $\mathbf{I}_{(\varphi \circ \alpha)(t)}\big(d\varphi_{\alpha(t)}(\alpha'(t))\big) > \mathbf{I}_{\alpha(t)}(\alpha'(t))$. This implies that the length of $(\varphi \circ \alpha)(J)$ is greater than the length of $\alpha(J)$; which is a contradiction. Therefore $\mathbf{I}_{\mathbf{p}}(\mathbf{v}) = \mathbf{I}_{\varphi(\mathbf{p})}\big(d\varphi_{\mathbf{p}}(\mathbf{v})\big)$, and we conclude that φ is an isometry.

4. The stereographic projection $\mathbf{x} : \mathbb{R}^2 \to \mathbb{S}^2$ is given by

$$\mathbf{x}(u, v) = \left(\frac{4u}{u^2 + v^2 + 4}, \frac{4v}{u^2 + v^2 + 4}, \frac{2(u^2 + v^2)}{u^2 + v^2 + 4} \right).$$

Then

$$\mathbf{x}_u = \left(\frac{-4u^2 + 4v^2 + 16}{(u^2 + v^2 + 4)^2}, \frac{-8uv}{(u^2 + v^2 + 4)^2}, \frac{16u}{(u^2 + v^2 + 4)^2} \right),$$

$$\mathbf{x}_v = \left(\frac{-8uv}{(u^2 + v^2 + 4)^2}, \frac{4u^2 - 4v^2 + 16}{(u^2 + v^2 + 4)^2}, \frac{16v}{(u^2 + v^2 + 4)^2} \right).$$

Therefore

$$E = \frac{16}{(u^2 + v^2 + 4)^2}, \quad F = 0, \quad G = \frac{16}{(u^2 + v^2 + 4)^2}.$$

Since $E = \lambda^2 \cdot 1$, $F = \lambda^2 \cdot 0$, $G = \lambda^2 \cdot 1$ for $\lambda = 4/(u^2 + v^2 + 4) > 0$, the sphere is locally conformal to a plane by Prop. 2.

5. Since $(\mathbf{x}_1)_s = \alpha_1'(s) + v\alpha_1''(s)$ and $(\mathbf{x}_1)_v = \alpha_1'(s)$,

$$E_1 = \langle \alpha_1'(s) + v\alpha_1''(s), \alpha_1'(s) + v\alpha_1''(s) \rangle = |\alpha_1'(s)|^2 + 2\alpha_1'(s) \cdot v\alpha_1''(s) + v^2 |\alpha_1''(s)|^2 = 1 + v^2 k_1(s)^2,$$

$$F_1 = \langle \alpha_1'(s) + v\alpha_1''(s), \alpha_1'(s) \rangle = |\alpha_1'(s)|^2 + v\alpha_1'(s) \cdot \alpha_1''(s) = 1,$$

$$G_1 = \langle \alpha_1'(s), \alpha_1'(s) \rangle = 1.$$

Similarly, $E_2 = 1 + v^2 k_2(s)$, $F_2 = 1$, $G_2 = 1$. The condition $k_1(s) = k_2(s) \neq 0$ implies that

$$E_1 = E_2, \quad F_1 = F_2, \quad G_1 = G_2.$$

Therefore $\mathbf{x}_1 \circ \mathbf{x}_2^{-1} : \mathbf{x}_2(V) \to \mathbf{x}_1(V)$ is a local isometry. Furthermore, since $\mathbf{x}_1(V)$ and $\mathbf{x}_2(V)$ are regular surfaces, both $\mathbf{x}_1|_V : V \to \mathbf{x}_1(V)$ and $\mathbf{x}_2|_V = V \to \mathbf{x}_2(V)$ are diffeomorphisms. So its composition $\mathbf{x}_1 \circ \mathbf{x}_2^{-1}$ is also diffeomorphic. We conclude that $\mathbf{x}_1 \circ \mathbf{x}_2^{-1}$ is an isometry.

6. We can assume that the curve $\alpha : I \to \mathbb{R}^3$ is parametrized by an arc length. The parametrization $\mathbf{x} : U \to \mathbb{R}^3$ of the tangent surface is given by $\mathbf{x}(t, v) = \alpha(t) + v\alpha'(t)$, where $U = \{(t, v) \in I \times \mathbb{R} \mid v \neq 0\}$.
Then by Prop. 2 of Sec. 2-3, for each $\mathbf{q} = (t_0, v_0) \in I \times (\mathbb{R} - \{\mathbf{0}\}) = U$, there exists a neighborhood V of $\mathbf{q}$ such that $\mathbf{x}(V)$ is a regular surface. Now we define

$$\beta(t) = \left(\int \cos \theta(t) \, dt, \int \sin \theta(t) \, dt, 0 \right)$$

for $t \in I$, where

$$\theta(t) = \int k(t) \, dt.$$

Then $\beta : I \to \mathbb{R}^3$ is a parametrized plane curve having $k(t)$ as curvature by Exercise 9 of Sec. 1-5. Let $\mathbf{y}(t, v) = \beta(t) + v\beta'(t)$, $(t, v) \in I \times (\mathbb{R} - \{\mathbf{0}\})$ be a tangent surface of β. By Prop. 2 of Sec. 2-3, there exist neighborhoods V_1, V_2 of (t_0, v_0) such that $\mathbf{x}(V_1)$ and $\mathbf{y}(V_2)$ are regular surfaces. Let $V = V_1 \cap V_2$ then $\mathbf{x}(V)$ and $\mathbf{y}(V)$ are both regular surfaces. Since the curvatures of α and β coincide, by Exercise 5, $\mathbf{x}(V)$ is isometric to $\mathbf{y}(V)$, which is an open set of the plane.

7. (a) $\Rightarrow$ (b): For all $\mathbf{v} \in V$, $|\mathbf{F}(\mathbf{v})| = \sqrt{\langle \mathbf{F}(\mathbf{v}), \mathbf{F}(\mathbf{v}) \rangle} = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = |\mathbf{v}|$.

(b) $\Rightarrow$ (c): Suppose that $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is an orthonormal basis in V. Then $|\mathbf{F}(\mathbf{v}_i)| = |\mathbf{v}_i| = 1$, and

$$\langle \mathbf{F}(\mathbf{v}_i), \mathbf{F}(\mathbf{v}_j) \rangle = \frac{1}{2}\left(|\mathbf{F}(\mathbf{v}_i) + \mathbf{F}(\mathbf{v}_j)|^2 - |\mathbf{F}(\mathbf{v}_i)|^2 - |\mathbf{F}(\mathbf{v}_j)|^2 \right)$$

$$= \frac{1}{2}\left(|\mathbf{F}(\mathbf{v}_i + \mathbf{v}_j)|^2 - |\mathbf{F}(\mathbf{v}_i)|^2 - |\mathbf{F}(\mathbf{v}_j)|^2 \right)$$

$$= \frac{1}{2}\left(|\mathbf{v}_i + \mathbf{v}_j|^2 - |\mathbf{v}_i|^2 - |\mathbf{v}_j|^2 \right) = \langle \mathbf{v}_i, \mathbf{v}_j \rangle$$

implies that $\{\mathbf{F}(\mathbf{v}_1), ..., \mathbf{F}(\mathbf{v}_n)\}$ is an orthonormal set. The linear independence comes from

$$\sum_{i=1}^{n} c_i \mathbf{F}(\mathbf{v}_i) = 0 \implies \mathbf{F}\left(\sum_{i=1}^{n} c_i \mathbf{v}_i \right) = 0 \implies \left| \sum_{i=1}^{n} c_i \mathbf{v}_i \right| = \left| \mathbf{F}\left(\sum_{i=1}^{n} c_i \mathbf{v}_i \right) \right| = 0$$

$$\implies \sum_{i=1}^{n} c_i \mathbf{v}_i = 0 \implies c_i = 0 \ \text{ for all } 1 \le i \le n.$$

Note that $\mathbf{F}$ is one-to-one, because

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}(\mathbf{y}) \implies \mathbf{F}(\mathbf{x} - \mathbf{y}) = 0 \implies |\mathbf{x} - \mathbf{y}| = |\mathbf{F}(\mathbf{x} - y)| = 0 \implies \mathbf{x} = \mathbf{y}.$$

Since N and W are both n-dimensional vector spaces, $\mathbf{F}$ is an onto map. Then for every $\mathbf{w} \in W$,

$$\mathbf{w} = \mathbf{F}(\mathbf{v}) = \mathbf{F}\left(\sum_{i=1}^{n} c_i \mathbf{v}_i \right) = \sum_{i=1}^{n} c_i \mathbf{F}(\mathbf{v}_i),$$

hence this set spans W. Therefore $\{\mathbf{F}(\mathbf{v}_1), ..., \mathbf{F}(\mathbf{v}_n)\}$ is an orthonormal basis in W.

(c) $\Rightarrow$ (d): Since W is a n-dimensional vector space, there is a basis $\{\mathbf{w}_1, ..., \mathbf{w}_n\}$ of V. Then we can construct an orthonormal basis $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ from $\{\mathbf{w}_1, ..., \mathbf{w}_n\}$ by using the Gram–Schmidt process. By (c), we can conclude that $\{\mathbf{F}(\mathbf{v}_1), ..., \mathbf{F}(\mathbf{v}_n)\}$ is an orthonormal basis in W.

(d) $\Rightarrow$ (a): There exists an orthonormal basis $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ in V such that $\{\mathbf{F}(\mathbf{v}_1), ..., \mathbf{F}(\mathbf{v}_n)\}$ is an orthonormal basis in W. For arbitrary vector $\mathbf{x}, \mathbf{y} \in V$, there exist scalars $a_i, b_i \, (1 \le i \le n)$ such that

$$\mathbf{x} = \sum_{i=1}^{n} a_i \mathbf{v}_i, \quad \mathbf{y} = \sum_{i=1}^{n} b_i \mathbf{v}_i.$$

Then

$$\langle \mathbf{F}(\mathbf{x}), \mathbf{F}(\mathbf{y}) \rangle = \left\langle \mathbf{F}\left(\sum_{i=1}^{n} a_i \mathbf{v}_i \right), \mathbf{F}\left(\sum_{i=1}^{n} b_j \mathbf{v}_j \right) \right\rangle = \left\langle \sum_{i=1}^{n} a_i \mathbf{F}(\mathbf{v}_i), \sum_{j=1}^{n} b_j \mathbf{F}(\mathbf{v}_j) \right\rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j \langle \mathbf{F}(\mathbf{v}_i), \mathbf{F}(\mathbf{v}_j) \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j \mathbf{1}_{i=j} = \sum_{i=1}^{n} a_i b_i = \langle \mathbf{x}, \mathbf{y} \rangle.$$

8. Define $\mathbf{F}(\mathbf{p}) = \mathbf{G}(\mathbf{p}) - \mathbf{G}(\mathbf{0})$. Then $|\mathbf{F}(\mathbf{p})| = |\mathbf{G}(\mathbf{p}) - \mathbf{G}(\mathbf{0})| = |\mathbf{p} - \mathbf{0}| = |\mathbf{p}|$. Moreover, $\mathbf{F}$ preserves the inner product of $\mathbb{R}^3$ because

$$\langle \mathbf{F}(\mathbf{p}), \mathbf{F}(\mathbf{q}) \rangle = \frac{1}{2}(|\mathbf{F}(\mathbf{p})|^2 + |\mathbf{F}(\mathbf{q})|^2 - |\mathbf{F}(\mathbf{p}) - \mathbf{F}(\mathbf{q})|^2)$$

$$= \frac{1}{2}(|\mathbf{p}|^2 + |\mathbf{q}|^2 - |\mathbf{G}(\mathbf{p}) - \mathbf{G}(\mathbf{q})|^2)$$

$$= \frac{1}{2}(|\mathbf{p}|^2 + |\mathbf{q}|^2 - |\mathbf{p} - \mathbf{q}|^2) = \langle \mathbf{p}, \mathbf{q} \rangle$$

for every vector $\mathbf{p}, \mathbf{q} \in \mathbb{R}^3$. Thus it maps the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$ onto an orthonormal basis. For $\mathbf{p} = \sum_{i=1}^{3} a_i \mathbf{e}_i$, we have

$$\langle \mathbf{F}(\mathbf{p}), \mathbf{F}(\mathbf{e}_i) \rangle = \langle \mathbf{p}, \mathbf{e}_i \rangle = a_i,$$

so $\mathbf{F}(\mathbf{p}) = \sum_{i=1}^{3} a_i \mathbf{F}(\mathbf{e}_i)$. Hence $\mathbf{F}$ is a linear map, and by Exercise 7, $\mathbf{F}$ is a linear isometry. Finally we can represent $\mathbf{G}$ as

$$\mathbf{G}(\mathbf{p}) = \mathbf{F}(\mathbf{p}) + \mathbf{G}(\mathbf{0}) = \mathbf{F}(\mathbf{p}) + \mathbf{p}_0$$

for all $\mathbf{p} \in \mathbb{R}^3$.

9. **a.** Since $\varphi : S_1 \to S_2$ is an isometry, φ is a diffeomorphism and for all $\mathbf{p}_1 \in S_1$ and $\mathbf{w}_1, \mathbf{w}_2 \in T_{\mathbf{p}_1}(S_1)$,

$$\langle \mathbf{w}_1, \mathbf{w}_2 \rangle_{\mathbf{p}_1} = \left\langle d\varphi_{\mathbf{p}_1}(\mathbf{w}_1), d\varphi_{\mathbf{p}_1}(\mathbf{w}_2) \right\rangle_{\varphi(\mathbf{p}_1)}.$$

Observe that $\varphi^{-1} : S_2 \to S_1$ is also a diffeomorphism. From $\varphi^{-1} \circ \varphi = \mathrm{Id}_{S_1}$ and $\varphi \circ \varphi^{-1} = \mathrm{Id}_{S_2}$,

$$d(\varphi^{-1})_{\varphi(\mathbf{p}_1)} \circ d\varphi_{\mathbf{p}_1} = d\left(\mathrm{Id}_{S_1}\right)_{\mathbf{p}_1} = \mathrm{Id}_{T_{\mathbf{p}_1}(S_1)}, \quad d\varphi_{\varphi^{-1}(\mathbf{p}_2)} \circ d(\varphi^{-1})_{\mathbf{p}_2} = d\left(\mathrm{Id}_{S_2}\right)_{\mathbf{p}_2} = \mathrm{Id}_{T_{\mathbf{p}_2}(S_2)}$$

For every $\mathbf{p}_1 \in S_1$ and $\mathbf{p}_2 \in S_2$. Hence $d\varphi_{\mathbf{p}_1}$ is bijective for every $\mathbf{p}_1 \in S_1$. Now given $\mathbf{p}_2 \in S_2$ and $\mathbf{v}_1, \mathbf{v}_2 \in T_{\mathbf{p}_2}(S_2)$, find $\mathbf{w}_1, \mathbf{w}_2 \in T_{\mathbf{p}_1}(S_1)$ such that $\mathbf{v}_1 = d\varphi_{\mathbf{p}_1}(\mathbf{w}_1)$ and $\mathbf{v}_2 = d\varphi_{\mathbf{p}_1}(\mathbf{w}_2)$ where $\mathbf{p}_1 = \varphi^{-1}(\mathbf{p}_2) \in S_1$. Then

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle_{\mathbf{p}_2} = \left\langle d\varphi_{\mathbf{p}_1}(\mathbf{w}_1), d\varphi_{\mathbf{p}_1}(\mathbf{w}_2) \right\rangle_{\varphi(\mathbf{p}_1)} = \langle \mathbf{w}_1, \mathbf{w}_2 \rangle_{\mathbf{p}_1} = \left\langle d(\varphi^{-1})_{\mathbf{p}_2}(\mathbf{v}_1), d(\varphi^{-1})_{\mathbf{p}_2}(\mathbf{v}_2) \right\rangle_{\varphi^{-1}(\mathbf{p}_2)}.$$

Therefore $\varphi^{-1} : S_2 \to S_1$ is also an isometry.

b. Since $\varphi : S_1 \to S_2$, $\psi : S_2 \to S_3$ are isometries, φ and ψ are diffeomorphisms and

$$\langle \mathbf{w}_1, \mathbf{w}_2 \rangle_{\mathbf{p}_1} = \left\langle d\varphi_{\mathbf{p}_1}(\mathbf{w}_1), d\varphi_{\mathbf{p}_1}(\mathbf{w}_2) \right\rangle_{\varphi(\mathbf{p}_1)}, \quad \text{for all } \mathbf{p}_1 \in S_1, \ \mathbf{w}_1, \mathbf{w}_2 \in T_{\mathbf{p}_1}(S_1),$$

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle_{\mathbf{p}_2} = \left\langle d\psi_{\mathbf{p}_2}(\mathbf{v}_1), d\psi_{\mathbf{p}_2}(\mathbf{v}_2) \right\rangle_{\varphi(\mathbf{p}_2)}, \quad \text{for all } \mathbf{p}_2 \in S_2, \ \mathbf{v}_1, \mathbf{v}_2 \in T_{\mathbf{p}_2}(S_2).$$

Given $\mathbf{p}_1 \in S_1$ and $\mathbf{w}_1, \mathbf{w}_2 \in T_{\mathbf{p}_1}(S_1)$, let $\mathbf{v}_1 = d\varphi_{\mathbf{p}_1}(\mathbf{w}_1)$, $\mathbf{v}_2 = d\varphi_{\mathbf{p}_1}(\mathbf{w}_2)$, and $\mathbf{p}_2 = \varphi(\mathbf{p}_1)$, then

$$\left\langle d(\psi \circ \varphi)_{\mathbf{p}_1}(\mathbf{w}_1), d(\psi \circ \varphi)_{\mathbf{p}_1}(\mathbf{w}_2) \right\rangle_{(\psi \circ \varphi)(\mathbf{p}_1)} = \left\langle \left(d\psi_{\varphi(\mathbf{p}_1)} \circ d\varphi_{\mathbf{p}_1}\right)(\mathbf{w}_1), \left(d\psi_{\varphi(\mathbf{p}_1)} \circ d\varphi_{\mathbf{p}_1}\right)(\mathbf{w}_2) \right\rangle_{\psi(\mathbf{p}_2)}$$

$$= \left\langle d\psi_{\mathbf{p}_2}(\mathbf{v}_1), d\psi_{\mathbf{p}_2}(\mathbf{v}_2) \right\rangle_{\psi(\mathbf{p}_2)} = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle_{\mathbf{p}_2}$$

$$= \left\langle d\varphi_{\mathbf{p}_1}(\mathbf{w}_1), d\varphi_{\mathbf{p}_1}(\mathbf{w}_2) \right\rangle_{\varphi(\mathbf{p}_1)} = \langle \mathbf{w}_1, \mathbf{w}_2 \rangle_{\mathbf{p}_1}.$$

Hence $\psi \circ \varphi : S_1 \to S_3$ is also an isometry.

10. Let $\varphi : S \to S$ denote the rotation by a specific angle about its axis. We suppose that $\hat{\mathbf{z}}$ is the axis of S. By Exercise 11 of Sec. 2-3, φ is a diffeomorphism of S. It is possible to extend $\varphi : S \to S$ to a linear map $\varphi^* : \mathbb{R}^3 \to \mathbb{R}^3$, which is the rotation by the same angle with φ. By Exercise 8 of Sec. 2-4,

$$d\varphi_{\mathbf{p}}(\mathbf{w}) = \varphi(\mathbf{w}), \quad \mathbf{p} \in S, \ w \in T_{\mathbf{p}}(S).$$

Moreover, since $|\varphi(\mathbf{p})| = |\mathbf{p}|$ for every $\mathbf{p} \in S$, φ is a linear isometry. Hence

$$\left\langle d\varphi_{\mathbf{p}}(\mathbf{w}_1), d\varphi_{\mathbf{p}}(\mathbf{w}_2) \right\rangle = \langle \varphi(\mathbf{w}_1), \varphi(\mathbf{w}_2) \rangle = \langle \mathbf{w}_1, \mathbf{w}_2 \rangle$$

for every $\mathbf{p} \in S$, $\mathbf{w}_1, \mathbf{w}_2 \in T_{\mathbf{p}}(S)$. Therefore φ is an isometry.

11. **a.** Since $\mathbf{F}$ is a distance-preserving map, there exists $\mathbf{p}_0 \in \mathbb{R}^3$ and a linear isometry $\mathbf{G}$ of the vector space $\mathbb{R}^3$ such that

$$\mathbf{F}(\mathbf{p}) = \mathbf{G}(\mathbf{p}) + \mathbf{p}_0, \quad \text{for all } \mathbf{p} \in \mathbb{R}^3$$

by Exercise 8. Now $\mathbf{F}|_S : S \to S$ is a restriction of diffeomorphism into a regular surface, so it is also a diffeomorphism. Note that

$$d\mathbf{F}_{\mathbf{p}}(\mathbf{w}) = d\mathbf{G}_{\mathbf{p}}(\mathbf{w}) = \mathbf{G}(\mathbf{w})$$

since $\mathbf{G}$ is linear. Then for every $\mathbf{p} \in S$ and $\mathbf{w}_1, \mathbf{w}_2 \in T_{\mathbf{p}}(S)$, we have

$$\left\langle d\mathbf{F}_{\mathbf{p}}(\mathbf{w}_1), d\mathbf{F}_{\mathbf{p}}(\mathbf{w}_2) \right\rangle = \langle \mathbf{G}(\mathbf{w}_1), \mathbf{G}(\mathbf{w}_2) \rangle = \langle \mathbf{w}_1, \mathbf{w}_2 \rangle.$$

Therefore $F|_S$ is an isometry of S.

b. Let $\mathbf{F}$ be an orthogonal linear transformations of $\mathbb{R}^3$. by Exercise 7(b), we have $|\mathbf{F}(x)| = |x|$. Hence

$$|\mathbf{F}(x) - \mathbf{F}(y)| = |\mathbf{F}(x - y)| = |x - y|$$

implies that $\mathbf{F}$ is a distance-preserving map. Then $\mathbf{F}$ becomes a diffeomorphism, since $\mathbf{F}$ can be represented as $\mathbf{F}(\mathbf{p}) = \mathbf{G}(\mathbf{p}) + \mathbf{p}_0$ for some point $\mathbf{p}_0 \in \mathbb{R}^3$ and linear isometry $\mathbf{G}$. Finally, since $|\mathbf{F}(\mathbf{x})| = |\mathbf{x}|$, we have $\mathbf{F}(S) \subset S$ for the unit sphere $S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$, thus $\mathbf{F}$ is an isomtry of S by (a).

c. We consider an isometry $\varphi : S_1 \to S_2$ defined as $\varphi(u, v) = (\cos u, \sin u, v)$ between a plane $(0, 2\pi) \times (0, 1)$ and a cylinder $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ 0 < z < 1\}$. Let $\mathbf{p} = (\pi/2, 1/2)$ and $\mathbf{q} = (3\pi/2, 1/2)$ then we have

$$|\varphi(\mathbf{p}) - \varphi(\mathbf{q})| = \left| \left(1, 0, \frac{1}{2}\right) - \left(-1, 0, -\frac{1}{2}\right) \right| = 2 < \pi = |\mathbf{p} - \mathbf{q}|.$$

Therefore φ cannot be extended into distance-preserving map $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$.
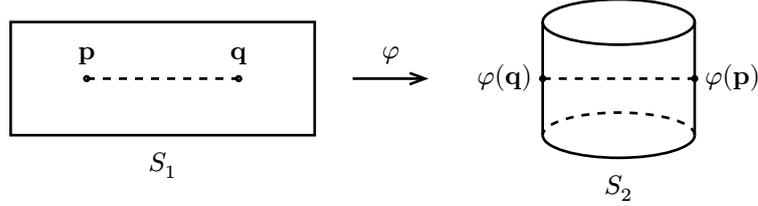
12. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ as $\mathbf{F}(x, y, z) = (x, -y, -z)$. Then $\mathbf{F}$ is a distance-preserving diffeomorphism of $\mathbb{R}^3$ because

$$|\mathbf{F}(x, y, z) - \mathbf{F}(x', y', z')| = |(x, -y, -z) - (x', -y', -z')| = \sqrt{(x - x')^2 + (-y + y')^2 + (-z + z')^2}$$

$$= \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} = |(x, y, z) - (x', y', z')|.$$

Since $\mathbf{F}(S) \subset S$, the restriction $\varphi = \mathbf{F}|_C$ is an isometry of C by Exercise 11(a). The fixed points of φ satisfies $(x, y, z) = \varphi(x, y, z) = (x, -y, -z)$, there are only two such points $(1, 0, 0)$ and $(-1, 0, 0)$.

13. (a) $\Rightarrow$ (b): For all $\mathbf{v} \in V$, $|\mathbf{G}(\mathbf{v})| = \sqrt{\langle \mathbf{G}(\mathbf{v}), \mathbf{G}(\mathbf{v}) \rangle} = \sqrt{\lambda^2 \langle \mathbf{v}, \mathbf{v} \rangle} = |\lambda| |\mathbf{v}|$, where $|\lambda| > 0$ is a constant.

(b) $\Rightarrow$ (c): Since W is a n-dimensional vector space, there is a basis $\{\mathbf{w}_1, ..., \mathbf{w}_n\}$ of V. Then we can construct an orthonormal basis $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ from $\{\mathbf{w}_1, ..., \mathbf{w}_n\}$ by using the Gram–Schmidt process. Then $\{\mathbf{G}(\mathbf{v}_1), ..., \mathbf{G}(\mathbf{v}_n)\}$ is ab orthogonal set because

$$\langle \mathbf{G}(\mathbf{v}_i), \mathbf{G}(\mathbf{v}_j) \rangle = \frac{1}{2} \left(|\mathbf{G}(\mathbf{v}_i) + \mathbf{G}(\mathbf{v}_j)|^2 - |\mathbf{G}(\mathbf{v}_i)|^2 - |\mathbf{G}(\mathbf{v}_j)|^2 \right)$$

$$= \frac{1}{2} \left(|\mathbf{G}(\mathbf{v}_i + \mathbf{v}_j)|^2 - |\mathbf{G}(\mathbf{v}_i)|^2 - |\mathbf{G}(\mathbf{v}_j)|^2 \right)$$

$$= \frac{1}{2} \left(\lambda^2 |\mathbf{v}_i + \mathbf{v}_j|^2 - \lambda^2 |\mathbf{v}_i|^2 - \lambda^2 |\mathbf{v}_j|^2 \right) = \lambda^2 \langle \mathbf{v}_i, \mathbf{v}_j \rangle.$$

Now we assert that $\{\mathbf{G}(\mathbf{v}_1), ..., \mathbf{G}(\mathbf{v}_n)\}$ is a basis of W. Linearly independence can be shown by

$$\sum_{i=1}^n c_i \mathbf{G}(\mathbf{v}_i) = 0 \implies \mathbf{G}\left(\sum_{i=1}^n c_i \mathbf{v}_i \right) = 0 \implies \left| \sum_{i=1}^n c_i \mathbf{v}_i \right| = \frac{1}{\lambda} \left| \mathbf{G}\left(\sum_{i=1}^n c_i \mathbf{v}_i \right) \right| = 0$$

$$\implies \sum_{i=1}^n c_i \mathbf{v}_i = 0 \implies c_i = 0 \ \text{ for all } 1 \le i \le n.$$

Note that $\mathbf{G}$ is one-to-one, because

$$\mathbf{G}(\mathbf{x}) = \mathbf{G}(\mathbf{y}) \implies \mathbf{G}(\mathbf{x} - \mathbf{y}) = 0 \implies |\mathbf{x} - \mathbf{y}| = \frac{1}{\lambda} |\mathbf{G}(\mathbf{x} - \mathbf{y})| = 0 \implies \mathbf{x} = \mathbf{y}.$$

Since N and W are both n-dimensional vector spaces, $\mathbf{G}$ is an onto map. Then for every $\mathbf{w} \in W$,

$$\mathbf{w} = \mathbf{G}(\mathbf{v}) = \mathbf{G}\left(\sum_{i=1}^n c_i \mathbf{v}_i \right) = \sum_{i=1}^n c_i \mathbf{G}(\mathbf{v}_i),$$

hence this set spans W. Therefore $\{\mathbf{G}(\mathbf{v}_1), ..., \mathbf{G}(\mathbf{v}_n)\}$ is an orthogonal basis in W. Note that

$$|\mathbf{G}(\mathbf{v}_i)| = \lambda |\mathbf{v}_i| = \lambda,$$

so the vectors $\mathbf{G}(\mathbf{v}_i)$, $1 \leq i \leq n$, have the same nonzero length.

(c) $\Rightarrow$ (a): There exists an orthonormal basis $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ in V such that $\{\mathbf{G}(\mathbf{v}_1), ..., \mathbf{G}(\mathbf{v}_n)\}$ is an orthogonal basis in W. For arbitrary vector $\mathbf{x}, \mathbf{y} \in V$, there exist scalars a_i, b_i $(1 \leq i \leq n)$ such that

$$\mathbf{x} = \sum_{i=1}^{n} a_i \mathbf{v}_i, \quad \mathbf{y} = \sum_{i=1}^{n} b_i \mathbf{v}_i.$$

Let $\lambda > 0$ be the same nonzero length of $\mathbf{G}(\mathbf{v}_i)$, $1 \leq i \leq n$. Then

$$\langle \mathbf{G}(\mathbf{x}), \mathbf{G}(\mathbf{y}) \rangle = \left\langle \mathbf{G}\left(\sum_{i=1}^{n} a_i \mathbf{v}_i\right), \mathbf{G}\left(\sum_{i=1}^{n} b_j \mathbf{v}_j\right) \right\rangle = \left\langle \sum_{i=1}^{n} a_i \mathbf{G}(\mathbf{v}_i), \sum_{j=1}^{n} b_j \mathbf{G}(\mathbf{v}_j) \right\rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j \langle \mathbf{G}(\mathbf{v}_i), \mathbf{G}(\mathbf{v}_j) \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j \lambda^2 \mathbf{1}_{i=j} = \lambda^2 \sum_{i=1}^{n} a_i b_i = \lambda^2 \langle \mathbf{x}, \mathbf{y} \rangle.$$

14. First suppose that φ is locally conformal. Then for every $\mathbf{p} \in S_1$, there exists a neighborhood V of $\mathbf{p}$ and $\overline{V}$ of $\varphi(\mathbf{p})$ such that $\varphi|_V : V \to \overline{V}$ is a conformal map. Then for every $\mathbf{p} \in V$ and $\mathbf{v}_1, \mathbf{v}_2 \in T_\mathbf{p}(S_1)$,

$$\langle d\varphi_\mathbf{p}(\mathbf{v}_1), d\varphi_\mathbf{p}(\mathbf{v}_2) \rangle = \lambda^2(\mathbf{p}) \langle \mathbf{v}_1, \mathbf{v}_2 \rangle_\mathbf{p}.$$

Thus

$$\cos(d\varphi_\mathbf{p}(\mathbf{v}_1), d\varphi_\mathbf{p}(\mathbf{v}_2)) = \frac{\langle d\varphi_\mathbf{p}(\mathbf{v}_1), d\varphi_\mathbf{p}(\mathbf{v}_2) \rangle}{|d\varphi_\mathbf{p}(\mathbf{v}_1)| |d\varphi_\mathbf{p}(\mathbf{v}_2)|} = \frac{\lambda^2(\mathbf{p}) \langle \mathbf{v}_1, \mathbf{v}_2 \rangle_\mathbf{p}}{\lambda^2(\mathbf{p}) |\mathbf{v}_1| |\mathbf{v}_2|} = \cos(\mathbf{v}_1, \mathbf{v}_2),$$

so φ preserves angles.

Conversely, suppose that φ preserves angles. Given $\mathbf{p} \in S_1$ and given $\mathbf{v}_1, \mathbf{v}_2 \in T_\mathbf{p}(S_1)$, let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the orthonormal basis of $T_\mathbf{p}(S_1)$. Since $\mathbf{e}_1$ and $\mathbf{e}_2$ are perpendicular, $d\varphi_\mathbf{p}(\mathbf{e}_1)$ and $d\varphi_\mathbf{p}(\mathbf{e}_2)$ are also perpendicular. Since $d\varphi_\mathbf{p}$ is linear,

$$\langle \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2 \rangle = |\mathbf{e}_1|^2 - |\mathbf{e}_2|^2 = 1 - 1 = 0$$
$$\implies \cos(\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2) = 0$$
$$\implies \cos(d\varphi_\mathbf{p}(\mathbf{e}_1 + \mathbf{e}_2), d\varphi_\mathbf{p}(\mathbf{e}_1 - \mathbf{e}_2)) = 0$$
$$\implies \langle d\varphi_\mathbf{p}(\mathbf{e}_1 + \mathbf{e}_2), d\varphi_\mathbf{p}(\mathbf{e}_1 - \mathbf{e}_2) \rangle = |d\varphi_\mathbf{p}(\mathbf{e}_1)|^2 - |d\varphi_\mathbf{p}(\mathbf{e}_2)|^2 = 0.$$

Hence $|d\varphi_\mathbf{p}(\mathbf{e}_1)| = |d\varphi_\mathbf{p}(\mathbf{e}_2)|$, and we define $\lambda(\mathbf{p}) > 0$ as this common value. Then for every point $\mathbf{v} = a\mathbf{e}_1 + b\mathbf{e}_2 \in T_\mathbf{p}(S_1)$,

$$|d\varphi_\mathbf{p}(\mathbf{v})| = |a \cdot d\varphi_\mathbf{p}(\mathbf{e}_1) + b \cdot d\varphi_\mathbf{p}(\mathbf{e}_2)|$$

$$= \sqrt{a^2 |d\varphi_\mathbf{p}(\mathbf{e}_1)|^2 + b^2 |d\varphi_\mathbf{p}(\mathbf{e}_2)|^2}$$

$$= |d\varphi_\mathbf{p}(\mathbf{e}_1)| \sqrt{a^2 + b^2} = \lambda(\mathbf{p}) |\mathbf{v}|.$$

Therefore for every $\mathbf{p} \in S_1$ and $\mathbf{v}_1, \mathbf{v}_2 \in T_\mathbf{p}(S_1)$,

$$\langle d\varphi_\mathbf{p}(\mathbf{v}_1), d\varphi_\mathbf{p}(\mathbf{v}_2) \rangle = \frac{|d\varphi_\mathbf{p}(\mathbf{v}_1)| |d\varphi_\mathbf{p}(\mathbf{v}_2)|}{|\mathbf{v}_1| |\mathbf{v}_2|} \langle \mathbf{v}_1, \mathbf{v}_2 \rangle_\mathbf{p} = \lambda(\mathbf{p})^2 \langle \mathbf{v}_1, \mathbf{v}_2 \rangle_\mathbf{p},$$

so φ is locally conformal.

15. Fix $\mathbf{p} \in \mathbb{R}^2 - Q$. Given $\mathbf{w}_1 \in \mathbb{R}^2$, there exists a parametrized curve $\alpha_1 : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ such that

$$\mathbf{w}_1 = \alpha_1'(0) = (x_1'(0), y_1'(0)).$$

Then

$$d\varphi_\mathbf{p}(\mathbf{w}_1) = (\varphi \circ \alpha_1)'(0) = \frac{d}{dt}(\mathbf{u}(x_1(t), y_1(t)), \mathbf{v}(x_1(t), y_1(t)))\Big|_{t=0}$$

$$= \left(\mathbf{u}_x x_1'(0) + \mathbf{u}_y y_1'(0), \mathbf{v}_x x_1'(0) + \mathbf{v}_y y_1'(0)\right).$$

Similarly, $\mathbf{w}_2 = (x_2'(0), y_2'(0))$ then $d\varphi_\mathbf{p}(\mathbf{w}_2) = \left(\mathbf{u}_x x_2'(0) + \mathbf{u}_y y_2'(0), \mathbf{v}_x x_2'(0) + \mathbf{v}_y y_2'(0)\right)$. Therefore

$$\langle d\varphi_{\mathbf{p}}(\mathbf{w}_1), d\varphi_{\mathbf{p}}(\mathbf{w}_2)\rangle = \langle (\mathbf{u}_x x_1' + \mathbf{u}_y y_1', \mathbf{v}_x x_1' + \mathbf{v}_y y_1'), (\mathbf{u}_x x_2' + \mathbf{u}_y y_2', \mathbf{v}_x x_2' + \mathbf{v}_y y_2')\rangle$$

$$= \mathbf{u}_x^2 x_1' x_2' + \mathbf{u}_x \mathbf{u}_y (x_1' y_2' + y_1' x_2') + \mathbf{u}_y^2 y_1' y_2'$$

$$+ \mathbf{v}_x^2 x_1' x_2' + \mathbf{v}_x \mathbf{v}_y (x_1' y_2' + y_1' x_2') + \mathbf{v}_y^2 y_1' y_2'$$

$$= (\mathbf{u}_x^2 + \mathbf{v}_x^2) x_1' x_2' + (\mathbf{u}_x \mathbf{u}_y + \mathbf{v}_x \mathbf{v}_y)(x_1' y_2' + y_1' x_2') + (\mathbf{u}_y^2 + \mathbf{v}_y^2) y_1' y_2'.$$

The Cauchy-Riemann equations $\mathbf{u}_x = \mathbf{v}_y$, $\mathbf{u}_y = -\mathbf{v}_x$ imply that

$$\mathbf{u}_x^2 = \mathbf{v}_y^2, \quad \mathbf{u}_x \mathbf{u}_y + \mathbf{v}_x \mathbf{v}_y = 0, \quad \mathbf{v}_x^2 = \mathbf{u}_y^2.$$

Therefore

$$\langle d\varphi_{\mathbf{p}}(\mathbf{w}_1), d\varphi_{\mathbf{p}}(\mathbf{w}_2)\rangle = (\mathbf{u}_x^2 + \mathbf{u}_y^2)(x_1' x_2' + y_1' y_2') = (\mathbf{u}_x^2 + \mathbf{u}_y^2)\langle \mathbf{w}_1, \mathbf{w}_2\rangle.$$

By letting $\lambda^2(\mathbf{p}) = (\mathbf{u}_x^2 + \mathbf{u}_y^2) > 0$, we conclude that φ is a local conformal map from $\mathbb{R}^2 - Q$ to $\mathbb{R}$.

16. This change of parameters $u = \log \tan \theta/2$ and $v = \varphi$ is possible since the map is one-to-one and the Jacobian

$$\frac{\partial(u, v)}{\partial(\theta, \varphi)} = \csc \theta > 0$$

is nonzero everywhere. $\tan \theta/2 = e^u$ implies

$$\cos \theta = \frac{1 - \tan^2 \theta/2}{1 + \tan^2 \theta/2} = \frac{1 - e^{2u}}{1 + e^{2u}} = -\frac{e^u - e^{-u}}{e^u + e^{-u}} = -\tanh u,$$

$$\sin \theta = \frac{2 \tan \theta/2}{1 + \tan^2 \theta/2} = \frac{2e^u}{1 + e^{2u}} = \frac{2}{e^u + e^{-u}} = \operatorname{sech} u.$$

Therefore the new parametrization of the coordinate neighborhood $\mathbf{x}(U) = V$ is given by

$$\mathbf{y}(u, v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, -\tanh u).$$

Since

$$\mathbf{y}_u = (-\operatorname{sech} u \tanh u \cos v, -\operatorname{sech} u \tanh u \sin v, -\operatorname{sech}^2 u), \quad \mathbf{y}_v = (-\operatorname{sech} u \sin v, \operatorname{sech} u \cos v, 0),$$

We obtain $E = \operatorname{sech}^2 u \tanh^2 u + \operatorname{sech}^4 u = \operatorname{sech}^2 u$, $F = 0$, $G = \operatorname{sech}^2 u$.

 Note We can change of parameters by $\overline{u} = -u$ to obtain the parametrization $\mathbf{y}$ given in the text.

17. The loxodromes make a constant angle with the meridians of the sphere. Under Mercator's projection, the meridians go into parallel straight lines in the plane. Since Mercator's projection is conformal, the loxodromes also go into straight lines. Recall that the angles are preserved by conformal maps. Thus the sum of the interior angles of the triangle in the sphere is the same as the sum of the interior angles of a rectilinear plane triangle, which is π.

18. Since φ is conformal, it is a diffeomorphism and for any $\mathbf{p} \in S$, $\mathbf{w}_1, \mathbf{w}_2 \in T_{\mathbf{p}}(S)$,

$$\langle d\varphi_{\mathbf{p}}(\mathbf{w}_1), d\varphi_{\mathbf{p}}(\mathbf{w}_2)\rangle_{\varphi(\mathbf{p})} = \lambda^2(\mathbf{p})\langle \mathbf{w}_1, \mathbf{w}_2\rangle_{\mathbf{p}}.$$

Fix $\mathbf{p} \in S$ and $\mathbf{x} : U \to S$ be the parametrization of S at $\mathbf{p}$, then $\varphi \circ \mathbf{x} : U \to \overline{S}$ is the parametrization of $\overline{S}$ at $\varphi(\mathbf{p})$. Since φ is area preserving, for any region $R \subset S$,

$$\iint_Q |\mathbf{x}_u \wedge \mathbf{x}_v| \, du \, dv = \iint_Q |(\varphi \circ \mathbf{x})_u \wedge (\varphi \circ \mathbf{x})_v| \, du \, dv, \quad Q = \mathbf{x}^{-1}(R).$$

Observe that

$$|(\varphi \circ \mathbf{x})_u \wedge (\varphi \circ \mathbf{x})_v|^2 = |d\varphi(\mathbf{x}_u) \wedge d\varphi(\mathbf{x}_v)|^2$$

$$= |d\varphi(\mathbf{x}_u)|^2 |d\varphi(\mathbf{x}_v)|^2 - \langle d\varphi(\mathbf{x}_u), d\varphi(\mathbf{x}_v)\rangle^2$$

$$= \lambda^4 |\mathbf{x}_u|^2 |\mathbf{x}_v|^2 - \lambda^4 \langle \mathbf{x}_u, \mathbf{x}_v\rangle^2 = \lambda^4 |\mathbf{x}_u \wedge \mathbf{x}_v|^2.$$

Therefore the equation

$$\iint_Q (\lambda^2 - 1)|\mathbf{x}_u \wedge \mathbf{x}_v|\, du\, dv = 0, \quad Q = \mathbf{x}^{-1}(R)$$

holds for any region R; hence we have $\lambda(\mathbf{p}) \equiv 1$ for every $\mathbf{p} \in S$. This implies that φ is an isometry.

19. It is easy to see that $\varphi : M \to C$ is given as

$$\varphi(x, y, z) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, z \right),$$

and it is a diffeomorphism. Let $\mathbf{x}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, $(0 < \theta < \pi, 0 < \varphi < 2\pi)$ be a parametrization for the sphere. Then we have

$$\mathbf{x}_\theta = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta), \ \mathbf{x}_\varphi = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)$$

so

$$E = 1, \ F = 0, \ G = \sin^2\theta, \ \sqrt{EG - F^2} = \sin\theta.$$

Observe that the map $\overline{\mathbf{x}}(\theta, \varphi) = (\cos\varphi, \sin\varphi, \cos\theta)$ given by $\overline{\mathbf{x}} = \varphi \circ \mathbf{x}$ is a parametrization for the circumscribed cylinder C. Moreover, we have

$$\overline{\mathbf{x}}_\theta = (0, 0, -\sin\theta), \ \overline{\mathbf{x}}_\varphi = (-\sin\varphi, \cos\varphi, 0) \implies \overline{E} = \sin^2\theta, \ \overline{F} = 0, \ \overline{G} = 1, \ \sqrt{\overline{EG} - \overline{F^2}} = \sin\theta.$$

Then φ is an area-preserving diffeomorphism because for any region $R \subset S$,

$$A(R) = \iint_Q \sqrt{EG - F^2}\, d\theta\, d\varphi = \iint_Q \sqrt{\overline{EG} - \overline{F^2}}\, d\theta\, d\varphi = A(\varphi(R)), \quad Q = \mathbf{x}^{-1}(R).$$

20. a. Observe that φ is differentiable. Given $\mathbf{q} = (u_0, v_0)$, we have

$$d\varphi_{\mathbf{q}} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\sqrt{f'(v_0)^2 + g'(v_0)^2}}{f(v_0)} \end{pmatrix}, \quad \det(d\varphi_{\mathbf{q}}) = \frac{\sqrt{f'(v_0)^2 + g'(v_0)^2}}{f(v_0)} > 0.$$

By inverse function theorem, there exists a neighborhood W_1 of $\mathbf{q}$ and W_2 of $\varphi(\mathbf{q})$ such that $\varphi|_{W_1} : W_1 \to W_2$ is a diffeomorphism. Hence φ is a local diffeomorphism.

b. We define a new parametrization $\overline{\mathbf{x}} : W_2 \subset \varphi(U) \to S$ for S as $\overline{\mathbf{x}} = \mathbf{x} \circ \varphi^{-1}$, that is,

$$(\overline{u}, \overline{v}) = \varphi(u, v) = \left(u, \int \frac{\sqrt{f'(v)^2 + g'(v)^2}}{f(v)}\, dv \right) \quad \text{and} \quad \overline{\mathbf{x}}(\overline{u}, \overline{v}) = \mathbf{x}(u, v).$$

Since

$$\overline{\mathbf{x}}_{\overline{u}} = (-f(v)\sin u, f(v)\cos u, 0), \quad \overline{\mathbf{x}}_{\overline{v}} = (f'(v)\cos u, f'(v)\sin u, g'(v)) \cdot \frac{dv}{d\overline{v}},$$

we have $E = f(v)^2, \ F = 0$, and

$$G = (f'(v)^2 + g'(v))^2 \left(\frac{dv}{d\overline{v}} \right)^2 = (f'(v)^2 + g'(v))^2 \left(\frac{\sqrt{f'(v)^2 + g'(v)^2}}{f(v)} \right)^{-2} = f(v)^2.$$

Therefore $\theta = \overline{\mathbf{x}}^{-1} : \mathbf{x}(W_1) \to W_2$ is a local conformal map. Note that θ takes the parallels and the meridians of the neighborhood V into an orthogonal system of straight lines in $\theta(V) \subset \mathbb{R}^2$.

c. ψ is differentiable, and given $\mathbf{q} = (u_0, v_0)$, we have

$$d\psi_{\mathbf{q}} = \begin{pmatrix} 1 & 0 \\ 0 & f(v_0)\sqrt{f'(v_0)^2 + g'(v_0)^2} \end{pmatrix}, \quad \det(d\psi_{\mathbf{q}}) = f(v_0)\sqrt{f'(v_0)^2 + g'(v_0)^2} > 0.$$

By inverse function theorem, there exists a neighborhood W_1 of $\mathbf{q}$ and W_2 of $\psi(\mathbf{q})$ such that $\psi|_{W_1} : W_1 \to W_2$ is a diffeomorphism. Hence ψ is a local diffeomorphism.

d. Similar to part (c), we define a new parametrization $\overline{\mathbf{x}} : W_2 \subset \psi(U) \to S$ for S as $\overline{\mathbf{x}} = \mathbf{x} \circ \psi^{-1}$, that is,

$$(\bar{u}, \bar{v}) = \varphi(u, v) = \left(u, \int f(v)\sqrt{f'(v)^2 + g'(v)^2}\, dv\right) \quad \text{and} \quad \bar{\mathbf{x}}(\bar{u}, \bar{v}) = \mathbf{x}(u, v).$$

Since

$$\bar{\mathbf{x}}_{\bar{u}} = (-f(v)\sin u, f(v)\cos u, 0), \quad \bar{\mathbf{x}}_{\bar{v}} = (f'(v)\cos u, f'(v)\sin u, g'(v)) \cdot \frac{dv}{d\bar{v}},$$

we have $E = f(v)^2$, $F = 0$, and

$$G = (f'(v)^2 + g'(v))^2 \left(\frac{dv}{d\bar{v}}\right)^2 = (f'(v)^2 + g'(v))^2 \left(f(v)\sqrt{f'(v)^2 + g'(v)^2}\right)^{-2} = \frac{1}{f(v)^2}.$$

Then $\sqrt{EG - F^2} = 1$, thus $\bar{\theta} = \bar{\mathbf{x}}^{-1} : \mathbf{x}(W_1) \to W_2$ is an area-preserving map into a plane.

4-3. The Gauss Theorem and the Equations of Compatibility

1. Since $F = 0$, the Christoffel symbols are given by

$$\Gamma^1_{11} = \frac{E_u}{2E}, \quad \Gamma^2_{11} = -\frac{E_v}{2G}, \quad \Gamma^1_{12} = \frac{E_v}{2E}, \quad \Gamma^2_{12} = \frac{G_u}{2G}, \quad \Gamma^1_{22} = -\frac{G_u}{2E}, \quad \Gamma^2_{22} = \frac{G_v}{2G}.$$

Hence the Gaussian curvature K equals

$$K = -\frac{1}{E}\left((\Gamma^2_{12})_u - (\Gamma^2_{11})_v + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12}\right)$$

$$= -\frac{1}{E}\left(\frac{G_{uu}G - G_u^2}{2G^2} + \frac{E_{vv}G - E_vG_v}{2G^2} - \frac{E_v^2}{4EG} + \frac{G_u^2}{2G^2} + \frac{E_vG_v}{4G^2} - \frac{E_uG_u}{4EG}\right)$$

$$= -\frac{G_{uu}}{2EG} + \frac{G_u^2}{4EG^2} - \frac{E_{vv}}{2EG} + \frac{E_vG_v}{4EG^2} + \frac{E_v^2}{4EG} + \frac{E_uG_u}{4E^2G}.$$

Therefore

$$-\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\}$$

$$= -\frac{1}{2\sqrt{EG}}\left\{\frac{E_{vv}\sqrt{EG} - E_v\frac{E_vG + EG_v}{2\sqrt{EG}}}{EG} + \frac{G_{uu}\sqrt{EG} - G_u\frac{E_uG + EG_u}{2\sqrt{EG}}}{EG}\right\}$$

$$= -\frac{E_{vv}}{2EG} + \frac{E_v(E_vG + EG_v)}{4E^2G^2} - \frac{G_{uu}}{2EG} + \frac{G_u(E_uG + EG_u)}{4E^2G^2}$$

$$= -\frac{G_{uu}}{2EG} + \frac{G_u^2}{4EG^2} - \frac{E_{vv}}{2EG} + \frac{E_vG_v}{4EG^2} + \frac{E_v^2}{4EG} + \frac{E_uG_u}{4E^2G} = K.$$

2. Since $F = 0$, we can use Exercise 1.

$$K = -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\} = -\frac{1}{2\lambda}\left\{\left(\frac{\lambda_v}{\lambda}\right)_v + \left(\frac{\lambda_u}{\lambda}\right)_u\right\}$$

$$= -\frac{1}{2\lambda}\{(\log\lambda)_{uu} + (\log\lambda)_{vv}\} = -\frac{1}{2\lambda}\Delta(\log\lambda).$$

If $E = G = (u^2 + v^2 + c)^{-2}$ and $F = 0$, then $\log\lambda = -2\log(u^2 + v^2 + c)$, hence

$$K = -\frac{1}{2\lambda}\Delta(\log\lambda) = -\frac{(u^2 + v^2 + c)^2}{2}\left(-2 \cdot \left(\frac{2(-u^2 + v^2 + c)}{(u^2 + v^2 + c)^2} + \frac{2(u^2 - v^2 + c)}{(u^2 + v^2 + c)^2}\right)\right)$$

$$= (u^2 + v^2 + c)^2 \cdot \frac{4c}{(u^2 + v^2 + c)^2} = 4c.$$

3. First consider the surface $\mathbf{x}(u, v) = (u\cos v, u\sin v, \log u)$. Since

$$\mathbf{x}_u = \left(\cos v, \sin v, \frac{1}{u}\right), \ \mathbf{x}_v = (-u \sin v, u \cos v, 0),$$

$$\mathbf{x}_{uu} = \left(0, 0, -\frac{1}{u^2}\right), \ \mathbf{x}_{uv} = (-\sin v, \cos v, 0), \ \mathbf{x}_{vv} = (-u \cos v, -u \sin v, 0),$$

the coefficients of the first and second fundamental forms are

$$E = 1 + \frac{1}{u^2}, \ F = 0, \ G = u^2, \ \sqrt{EG - F^2} = \sqrt{u^2 + 1}, \ e = -\frac{1}{u\sqrt{u^2 + 1}}, \ f = 0, \ g = \frac{u}{\sqrt{u^2 + 1}}.$$

Therefore the Gaussian curvature of $\mathbf{x}(U)$ is

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-1/(u^2 + 1)}{u^2 + 1} = -\frac{1}{(u^2 + 1)^2}.$$

Similarly, the derivatives of the surface $\overline{\mathbf{x}}(u, v) = (u \cos v, u \sin v, v)$ are

$$\overline{\mathbf{x}}_u = (\cos v, \sin v, 0), \ \overline{\mathbf{x}}_v = (-u \sin v, u \cos v, 1),$$

$$\overline{\mathbf{x}}_{uu} = (0, 0, 0), \ \overline{\mathbf{x}}_{uv} = (-\sin v, \cos v, 0), \ \overline{\mathbf{x}}_{vv} = (-u \cos v, -u \sin v, 0),$$

the coefficients of the first and second fundamental forms are

$$\overline{E} = 1, \ \overline{F} = 0, \ \overline{G} = u^2 + 1, \ \sqrt{\overline{EG} - \overline{F}^2} = \sqrt{u^2 + 1}, \ \overline{e} = 0, \ \overline{f} = \frac{1}{\sqrt{u^2 + 1}}, \ \overline{g} = 0.$$

Hence the Gaussian curvature of $\overline{\mathbf{x}}(U)$ is

$$\overline{K} = \frac{\overline{eg} - \overline{f}^2}{\overline{EG} - \overline{F}^2} = \frac{-1/(u^2 + 1)}{u^2 + 1} = -\frac{1}{(u^2 + 1)^2}.$$

Therefore the surfaces $\mathbf{x}(U)$ and $\overline{\mathbf{x}}(U)$ have equal Gaussian curvature at the points $\mathbf{x}(u, v)$ and $\overline{\mathbf{x}}(u, v)$. However, $\overline{\mathbf{x}} \circ \mathbf{x}^{-1}$ is not an isometry because $(E, F, G) \neq \left(\overline{E}, \overline{F}, \overline{G}\right)$. This shows that the converse of the Gauss theorem is not true.

4. Let $\mathbb{S}_r^2$ be a sphere of radius r, and let P be a plane. Given a point $\mathbf{p} \in \mathbb{S}_r^2$, suppose that there exists a neighborhood V of $\mathbf{p}$ in $\mathbb{S}_r^2$ and $\varphi : V \to \varphi(V) \subset \mathbb{R}^2$ such that φ is an isometry into a plane. By Theorema Egregium, the Gaussian curvature $K(\mathbf{p})$ at $\mathbf{p} \in \mathbb{S}_r^2$ is equal to the Gaussian curvature $K(\varphi(\mathbf{p}))$ at $\varphi(\mathbf{p}) \in \mathbb{R}^2$. However, $K(\mathbf{p}) = 1/r^2$ and $K(\varphi(\mathbf{p})) = 0$ leads to a contradiction. Therefore no neighborhood of a point in a sphere may be isometrically mapped into a plane.

5. Since $E = G = 1$ and $F = \cos \theta$, we obtain three pairs of equations

$$\begin{cases} \Gamma_{11}^1 + \Gamma_{11}^2 \cos \theta = 0 \\ \Gamma_{11}^1 \cos \theta + \Gamma_{11}^2 = -\theta_u \sin \theta, \end{cases} \quad \begin{cases} \Gamma_{12}^1 + \Gamma_{12}^2 \cos \theta = 0 \\ \Gamma_{12}^1 \cos \theta + \Gamma_{12}^2 = 0, \end{cases} \quad \begin{cases} \Gamma_{22}^1 + \Gamma_{22}^2 \cos \theta = -\theta_v \sin \theta \\ \Gamma_{22}^1 \cos \theta + \Gamma_{22}^2 = 0. \end{cases}$$

Then we obtain

$$\Gamma_{11}^1 = \frac{\theta_u}{\tan \theta}, \quad \Gamma_{11}^2 = -\frac{\theta_u}{\sin \theta}, \quad \Gamma_{12}^1 = \Gamma_{12}^2 = 0, \quad \Gamma_{22}^1 = -\frac{\theta_v}{\sin \theta}, \quad \Gamma_{22}^2 = \frac{\theta_v}{\tan \theta}.$$

Therefore

$$K = -\frac{1}{E}\left(\left(\Gamma_{12}^2\right)_u - \left(\Gamma_{11}^2\right)_v + \Gamma_{12}^1 \Gamma_{11}^2 + \Gamma_{12}^2 \Gamma_{12}^2 - \Gamma_{11}^2 \Gamma_{22}^2 - \Gamma_{11}^1 \Gamma_{12}^2\right)$$

$$= \left(-\frac{\theta_u}{\sin \theta}\right)_v - \frac{\theta_u}{\sin \theta} \cdot \frac{\theta_v}{\tan \theta} = -\frac{\theta_{uv} \sin \theta - \theta_u \theta_v \cos \theta}{\sin^2 \theta} - \frac{\theta_u \theta_v \cos \theta}{\sin^2 \theta} = -\frac{\theta_{uv}}{\sin \theta}.$$

6. Suppose that there exists a surface $x(u, v)$ such that $E = G = 1$, $F = 0$ and $e = 1, g = -1, f = 0$. Then

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-1}{1} = -1.$$

On the other hand, since $F = 0$, we can apply the result of Exercise 1. Then we get

$$K = -\frac{1}{2}\{0 + 0\} = 0,$$

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which is a contradiction. Therefore there exists no such surface.

7. No. Suppose that there exists a surface $\mathbf{x}(u, v)$ with
$$E = 1, \ F = 0, \ G = \cos^2 u, \ e = \cos^2 u, \ f = 0, \ g = 1.$$

Then this surface must satisfy the Mainardi-Codazzi equations. Since $F = f = 0$, the Mainardi-Codazzi equations can be simplified into the following forms:
$$e_v = \frac{E_v}{2}\left(\frac{e}{E} + \frac{g}{G}\right), \quad g_u = \frac{G_u}{2}\left(\frac{e}{E} + \frac{g}{G}\right).$$

However, $g_u = 0$ and $\frac{G_u}{2}\left(\frac{e}{E} + \frac{g}{G}\right) = -\cos u \sin u \left(\cos^2 u + \frac{1}{\cos^2 u}\right)$, so $\mathbf{x}$ doesn't satisfy the Mainardi-Codazzi equations. Therefore there is no such surface.

8. a. Let $\mathbf{x}(u, v) = (u, v, 0)$ be the parametrization for the plane in Cartesian coordinates. Then $\mathbf{x}_{uu} = \mathbf{x}_{uv} = \mathbf{x}_{vv} = (0, 0, 0)$, therefore all the Christoffel symbols are zero by definition. It implies that $K = 0$.

b. Let $\mathbf{x}(\rho, \theta) = (\rho\cos\theta, \rho\sin\theta, 0)$ be the parametrization for the plane in polar coordinates. Then we have
$$\mathbf{x}_\rho = (\cos\theta, \sin\theta, 0), \quad \mathbf{x}_\theta = (-\rho\sin\theta, \rho\cos\theta, 0),$$
$$\mathbf{x}_{\rho\rho} = (0, 0, 0), \quad \mathbf{x}_{\rho\theta} = (-\sin\theta, \cos\theta, 0), \quad \mathbf{x}_{\theta\theta} = (-\rho\cos\theta, -\rho\sin\theta, 0).$$

Since
$$\mathbf{x}_{\rho\rho} = (0, 0, 0), \quad \mathbf{x}_{\rho\theta} = \frac{1}{\rho}\mathbf{x}_\theta, \quad \mathbf{x}_{\theta\theta} = -\rho\mathbf{x}_\rho,$$

we have $\Gamma_{11}^1 = \Gamma_{11}^2 = 0$, $\Gamma_{12}^1 = 0$, $\Gamma_{12}^2 = 1/\rho$, $\Gamma_{22}^1 = -\rho$, $\Gamma_{22}^2 = 0$. Then the Gaussian curvature is
$$K = -\frac{1}{E}\left((\Gamma_{12}^2)_\rho - (\Gamma_{11}^2)_\theta + \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{11}^1\Gamma_{12}^2\right) = -\left(-\frac{1}{\rho^2} + \left(\frac{1}{\rho}\right)^2\right) = 0.$$

9. By Theorema Egregium, if two surfaces are locally isometric, then the Gaussian curvature K of corresponding points must agree. However, the Gaussian curvatures of the given surfaces are
$$K_{\text{Sphere}} > 0, \quad K_{\text{Cylinder}} = 0, \quad K_{\text{Saddle}} < 0.$$

Therefore the surfaces are not pairwise locally isometric.

4-4. Parallel Transport. Geodesics.

1. a. Given any point $\mathbf{p} \in C$, since $C \subset S$ is a geodesic, there exists a parametrization $\alpha : I \to C$ of C at $\mathbf{p}$ such that α is a parametrized geodesic. Since C is a line of curvature, we have $\mathbf{N}' = \lambda\mathbf{t}$, and since C is a geodesic, we obtain $\mathbf{N} = \pm\mathbf{n}$. Then
$$\mathbf{N}' = \pm\mathbf{n}' = \pm(-k\mathbf{t} - \tau\mathbf{b}) = \lambda\mathbf{t}$$
implies that $\tau \equiv 0$ in the neighborhood of $\mathbf{p}$. Since $\mathbf{p} \in C$ is arbitrary, C is a plane curve.

b. Since the curve $C \subset S$ is a geodesic, $\mathbf{N} = \pm\mathbf{n}$. Then we have
$$\mathbf{N}' = \pm\mathbf{n}' = \pm(-k\mathbf{t} - \tau\mathbf{b}) = \mp k\mathbf{t},$$
where we applied $\tau \equiv 0$ because C is a plane curve. Therefore C is a line of curvature.

c. Consider the circle $C = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ on the plane $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$. Then C is a line of curvature because $\mathbf{N}' = \mathbf{0} = 0\mathbf{t}$ and a plane curve. However, C is not a geodesic, because the straight lines are the only geodesics of the plane.

2. If a curve $C \subset S$ is an asymptotic curve ($k_n = 0$) and a geodesic ($k_g = 0$), then we have
$$k^2 = k_n^2 + k_g^2 = 0$$

at every point $\mathbf{p} \in C$, which implies that C is a straight line. Conversely, if C is a straight line, then $k^2 = k_n^2 + k_g^2 = 0$ at every point $\mathbf{p} \in C$, therefore C is an asymptotic curve and a geodesic.

3. Given any point $\mathbf{p} \in C$, since $C \subset S$ is a geodesic, there exists a parametrization $\alpha : I \to C$ of C at $\mathbf{p}$ such that α is a parametrized geodesic. Note that $\mathbf{N} = \pm\mathbf{n}$ because C is a geodesic. Since C is a plane curve, we have $\mathbf{N}' \equiv 0$, therefore

$$\mathbf{N}' = \pm\mathbf{n}' = \pm(-k\mathbf{t} - \tau\mathbf{b}) \equiv 0.$$

Hence we obtain $k \equiv 0$ in the neighborhood of each point $\mathbf{p} \in C$, thus C is a straight line. We conclude that the straight lines are the only geodesics of a plane.
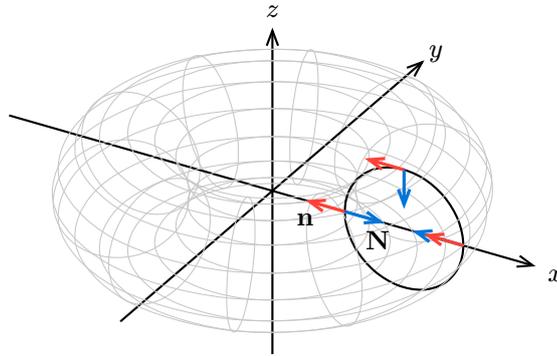
4. By definition, $\frac{d\mathbf{v}}{dt} - \frac{D\mathbf{v}}{dt}$ is perpendicular to the tangent plane $T_{\mathbf{p}}(S)$. Since $\mathbf{w}(t) \in T_{\mathbf{p}}(S)$, we obtain

$$\left\langle \frac{d\mathbf{v}}{dt} - \frac{D\mathbf{v}}{dt}, \mathbf{w} \right\rangle = 0 \implies \left\langle \frac{d\mathbf{v}}{dt}, \mathbf{w} \right\rangle = \left\langle \frac{D\mathbf{v}}{dt}, \mathbf{w} \right\rangle.$$

Similarly, we have $\left\langle \mathbf{v}, \frac{d\mathbf{w}}{dt} \right\rangle = \left\langle \mathbf{v}, \frac{D\mathbf{w}}{dt} \right\rangle$, therefore we obtain

$$\frac{d}{dt}\langle \mathbf{v}, \mathbf{w} \rangle = \left\langle \frac{d\mathbf{v}}{dt}, \mathbf{w} \right\rangle + \left\langle \mathbf{v}, \frac{d\mathbf{w}}{dt} \right\rangle = \left\langle \frac{D\mathbf{v}}{dt}, \mathbf{w} \right\rangle + \left\langle \mathbf{v}, \frac{D\mathbf{w}}{dt} \right\rangle.$$

5.

   a. A geodesic is a curve such that $\mathbf{N} = \pm\mathbf{n}$: the maximum parallel, and the minimum parallel.

   b. An asymptotic curve is a curve such that $k_n = 0$, that is, $\mathbf{N} \perp \mathbf{n}$: the upper parallel.

   c. A line of curvature is a curve such that $\mathbf{N}' = \lambda\mathbf{t}$: all the parallels presented.

6. Since $k_n = 0$ at the upper parallel and $k^2 = k_n^2 + k_g^2$, it follows that $k_g = \pm k = \pm 1/a$. Assuming the normal vector N points toward the interior of the torus, then we have $k_g = -1/a < 0$.

7. **a.** Note that if we intersect the cylinder $x^2 + y^2 = 1$ with a plane $z = x\tan\theta$, the intersecting curve C is given by

$$C = \{(x, y, y\tan\theta) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}.$$

If we take $r = y/(\cos\theta)$ in the $z = y\tan\theta$ plane, then we have $x^2 + r^2\cos^2\theta = 1$. Therefore intersecting curve is an ellipse on the plane.

   b. At the points where the curve C meets the x axis, unit normal vector $\mathbf{N}$ of cylinder and the normal vector $\mathbf{n}$ of C are parallel. Hence $k_g = 0$ at these points. Now we consider the other vertices. Since the angle between $\mathbf{N}$ and $\mathbf{n}$ is θ, we have that $k_g = k\sin\theta$, where k is the curvature of C. Since the curvature of ellipse at vertex is given by $k = \frac{1/(\cos\theta)}{1^2} = 1/(\cos\theta)$, we obtain $k_g = k\sin\theta = \tan\theta$.
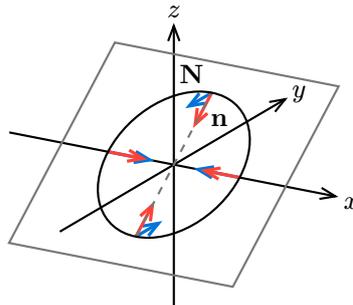
8. We shall show that all points of a given connected surface S are umbilical points, then by Prop. 4 of Sec. 3-2, S is either contained in a sphere or in a plane. Let $\mathbf{p} \in S$. For every $\mathbf{w} \in T_{\mathbf{p}}(S)$, $\mathbf{w} \neq \mathbf{0}$, there exists an $\varepsilon > 0$ and an unique parametrized geodesic $\gamma : (-\varepsilon, \varepsilon) \to S$ such that $\gamma(0) = \mathbf{p}$, $\gamma'(0) = \mathbf{w}$ by Prop. 5. Then $\mathbf{N} = \pm\mathbf{n}$ on the geodesic γ, and γ is a plane curve, so we have $\mathbf{N}' = \pm(-k\mathbf{t} - \tau\mathbf{b}) = \mp k\mathbf{t}$. Therefore γ is a line of curvature. Observe that every tangent vector $\mathbf{w} \in T_{\mathbf{p}}(S)$ is a principal direction. Hence the point $\mathbf{p}$ must be an umbilical point. Now we asserted that all points of a given surface are umbilical points, and therefore the surface is contained in a plane or a sphere.

9. Since the meridians are geodesics, the field of its tangent vectors is parallel along meridians. Because $\mathbf{w}_0$ is a tangent vector of C_1 at $\mathbf{p}_1$, $\mathbf{w}_1$ is also a tangent vector of C_1 at $\mathbf{p}_2$. On the other hand, the parallel transport preserves angles. Since the angle between the tangent vector of C_2 at $\mathbf{p}_1$ and $\mathbf{w}_0$ is φ, the angle between the tangent vector of C_2 at $\mathbf{p}_2$ and $\mathbf{w}_2$ is also φ. Then we conclude that the angle between $\mathbf{w}_1$ and $\mathbf{w}_2$ is 2φ.
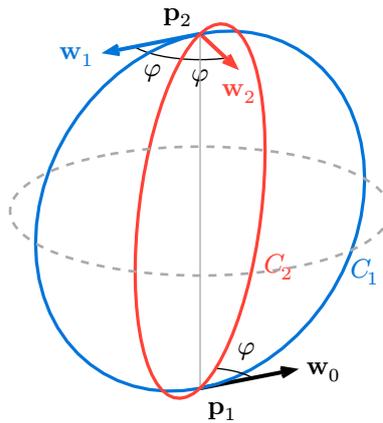

10. Let $\overline{C}$ be a plane curve obtained by projecting C onto the tangent plane $T_{\mathbf{p}}(S)$ along the normal to the surface at $\mathbf{p}$. Let $\alpha : (-\varepsilon, \varepsilon) \to C$ and $\overline{\alpha} : (-\varepsilon, \varepsilon) \to \overline{C}$ be the parametrizations at $\mathbf{p}$ of C and $\overline{C}$, respectively, such that $\alpha(0) = \overline{\alpha}(0) = \mathbf{p}$, then we have $\alpha'(0) = \overline{\alpha}'(0)$. We consider a projecting cylinder $\overline{S}$, which consists of the straight lines orthgonal to $T_{\mathbf{p}}(S)$ and intersect the curve C. Note that $\overline{S}$ contains both curves C and $\overline{C}$, and that $\overline{C}$ is the normal section of $\overline{S}$ along $\alpha'(0)$. By Meusnier's theorem, we have

$$k_n = \overline{k}_n$$

where k_n is the normal curvature of $C \subset \overline{S}$ at $\mathbf{p}$, and $\overline{k}_n$ is the normal curvature of $\overline{C} \subset \overline{S}$ at $\mathbf{p}$. Note that

$$k\mathbf{n} = k_g \overline{\mathbf{N}} + k_n \mathbf{N},$$

where $\mathbf{N}$ and $\overline{\mathbf{N}}$ is the unit normal vector of S and $\overline{S}$, respectively. It follows that $k_n = \left\langle k\mathbf{n}, \overline{\mathbf{N}} \right\rangle = k_g$. By definition, we have $\overline{k}_n = \overline{k} \cos 0 = \overline{k}$, where $\overline{k}$ is a curvature of $\overline{C}$ at $\mathbf{p}$. Therefore we obtain $k_g = \overline{k}$, as desired.
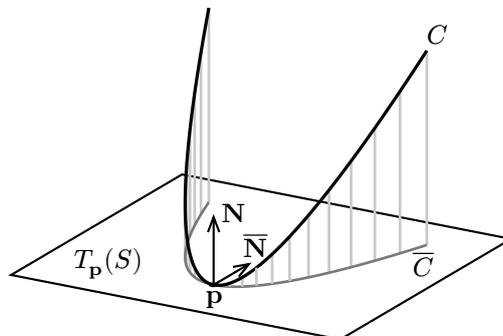
11. Let S and $\overline{S}$ be two oriented surfaces, and let $\varphi : S \to \overline{S}$ be an orientation-preserving isometry. Let $\mathbf{w}$ be a differential field of unit vectors along a parametrized curve $\alpha : I \to S$ on S. Let $\overline{\alpha} = \varphi \circ \alpha : I \to \overline{S}$ be

the corresponding curve on $\overline{S}$, and $\overline{\mathbf{w}} = d\varphi(\mathbf{w})$ be the corresponding vector field along $\overline{\alpha}$. We show that the algebraic value of the covariant derivative of $\mathbf{w}$ along α is equal to that of $\overline{\mathbf{w}}$ along $\overline{\alpha}$, that is,

$$\left[\frac{D\mathbf{w}}{dt}\right] = \left[\frac{D\overline{\mathbf{w}}}{dt}\right].$$

Since φ is an orientation-preserving isometry, we have

$$\langle d\varphi(\mathbf{w}_1), d\varphi(\mathbf{w}_2)\rangle = \langle \mathbf{w}_1, \mathbf{w}_2\rangle, \quad d\varphi(\mathbf{w}_1) \wedge d\varphi(\mathbf{w}_2) = d\varphi(\mathbf{w}_1 \wedge \mathbf{w}_2), \quad \mathbf{w}_1, \mathbf{w}_2 \in T_{\mathbf{p}}(S).$$

Note that $\overline{\mathbf{N}} = d\varphi(\mathbf{N})$ is the corresponding unit normal vector of $\overline{S}$, because for parametrization $\mathbf{x}(u,v)$ of S, we obtain

$$\overline{\mathbf{N}} = \frac{(\varphi \circ \mathbf{x})_u \wedge (\varphi \circ \mathbf{x})_v}{|(\varphi \circ \mathbf{x})_u \wedge (\varphi \circ \mathbf{x})_v|} = \frac{d\varphi(\mathbf{x}_u) \wedge d\varphi(\mathbf{x}_v)}{|d\varphi(\mathbf{x}_u) \wedge d\varphi(\mathbf{x}_v)|}$$

$$= \frac{d\varphi(\mathbf{x}_u \wedge \mathbf{x}_v)}{|d\varphi(\mathbf{x}_u \wedge \mathbf{x}_v)|} = \frac{d\varphi(\mathbf{x}_u \wedge \mathbf{x}_v)}{|\mathbf{x}_u \wedge \mathbf{x}_v|} = d\varphi\left(\frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}\right) = d\varphi(\mathbf{N}).$$

Therefore

$$\left[\frac{D\overline{\mathbf{w}}}{dt}\right] = \left\langle \frac{d\overline{\mathbf{w}}}{dt}, \overline{\mathbf{N}} \wedge \overline{\mathbf{w}}\right\rangle = \left\langle \frac{d(d\varphi(\mathbf{w}))}{dt}, d\varphi(\mathbf{N}) \wedge d\varphi(\mathbf{w})\right\rangle$$

$$= \left\langle d\varphi\left(\frac{d\mathbf{w}}{dt}\right), d\varphi(\mathbf{N} \wedge \mathbf{w})\right\rangle = \left\langle \frac{d\mathbf{w}}{dt}, \mathbf{N} \wedge \mathbf{w}\right\rangle = \left[\frac{D\mathbf{w}}{dt}\right].$$

12. We can parametrize a neighborhood of $\mathbf{p} \in S$ by $\mathbf{x}: U \to S$ in such a way that the two families of geodesics are the coordinate curves by Corollary 1 of Sec. 3-4. Since two differentiable families are orthogonal, we have $F = \langle \mathbf{x}_u, \mathbf{x}_v\rangle = 0$. Recall that the differential equations of the geodesics are given by

$$u'' + \Gamma_{11}^1(u')^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1(v')^2 = 0, \quad v'' + \Gamma_{11}^2(u')^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2(v')^2 = 0.$$

Since the curve obtained by $\alpha(t) = \mathbf{x}(u(t), \text{const.})$ is a geodesic, applying this to the second equation, we have $\Gamma_{11}^2 = 0$. Similarly, $\beta(t) = \mathbf{x}(\text{const.}, v(t))$ is a geodesic, it follows that $\Gamma_{22}^1 = 0$. Then

$$\Gamma_{22}^1 E + \Gamma_{22}^2 F = F_v - \frac{1}{2}G_u \implies G_u = 0,$$

$$\Gamma_{11}^1 F + \Gamma_{11}^2 G = F_u - \frac{1}{2}E_v \implies E_v = 0.$$

Therefore the Gaussian curvature of S is

$$K = -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\} = -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{0}{\sqrt{EG}}\right)_v + \left(\frac{0}{\sqrt{EG}}\right)_u\right\} = 0.$$

13. Fix two orthogonal unit vectors $\mathbf{v}(\mathbf{p})$ and $\mathbf{w}(\mathbf{p})$ in $T_{\mathbf{p}}(S)$. Then for each point $\mathbf{q} \in V$, we parallel transport $\mathbf{v}(\mathbf{p})$ and $\mathbf{w}(\mathbf{p})$ from $\mathbf{p}$ to $\mathbf{q}$ along an arbitrary curve. Since parallel transport between any two points of V does not depend on the curve joining these two points, we can define differentiable orthogonal unit vector fields $\mathbf{v}(\mathbf{q})$ and $\mathbf{w}(\mathbf{q})$. By Corollary 1 of Sec. 3-4, we can construct parametrization $\mathbf{x}$ for V such that the directions of the vector fields are tangent to the coordinate curves. Hence these are geodesics by definition. Now we can apply Exercise 12 to conclude that the Gaussian curvature is zero.

14. Since $|\mathbf{T}| = 1$, we have $\langle d\mathbf{T}/ds, \mathbf{T}\rangle = 0$. Therefore we can write

$$\frac{d\mathbf{T}}{ds} = \mathbf{0} + a\mathbf{V} + b\mathbf{N}$$

for some differentiable functions $a(s)$ and $b(s)$. Next, $|\mathbf{N}| = 1$ implies $\langle d\mathbf{N}/ds, \mathbf{N}\rangle = 0$, and $\langle \mathbf{N}, \mathbf{T}\rangle = 0$ implies that $\langle d\mathbf{N}/ds, \mathbf{T}\rangle = -\langle \mathbf{N}, d\mathbf{T}/ds\rangle = -b$. Therefore

$$\frac{d\mathbf{N}}{ds} = -b\mathbf{T} - c\mathbf{V} + \mathbf{0}$$

for some differentiable function $c(s)$. Finally, since $\mathbf{V} = \mathbf{N} \wedge \mathbf{T}$, we obtain

$$\frac{d\mathbf{V}}{ds} = \frac{d\mathbf{N}}{ds} \wedge \mathbf{T} + \mathbf{N} \wedge \frac{d\mathbf{T}}{ds} = (-b\mathbf{T} - c\mathbf{V}) \wedge \mathbf{T} + \mathbf{N} \wedge (a\mathbf{V} + b\mathbf{N}) = -a\mathbf{T} + \mathbf{0} + c\mathbf{N}.$$

a. Since $\langle d\mathbf{N}/ds, \mathbf{V} \rangle = -c$, we have $c = -\langle d\mathbf{N}/ds, \mathbf{V} \rangle$. Note that $\alpha(I) \subset S$ is a line of curvature if and only if $\mathbf{N}' = \lambda \mathbf{T}$, which is equivalent to $c \equiv 0$.

b. $b = \langle d\mathbf{T}/ds, \mathbf{N} \rangle = \langle \alpha'', \mathbf{N} \rangle = k\langle \mathbf{n}, \mathbf{N} \rangle$, which is a normal curvature of $\alpha(I) \subset S$ at $\mathbf{p}$.

c. $a = \langle d\mathbf{T}/ds, \mathbf{V} \rangle = \langle \alpha'', V \rangle = k\langle \mathbf{n}, \mathbf{N} \wedge \mathbf{t} \rangle$, which is a geodesic curvature of $\alpha(I) \subset S$ at $\mathbf{p}$.

15. a. Remember that the meridians and parallels of the sphere are geodesics, so the field of its tangent vectors is parallel along meridians. Since $\mathbf{v}$ is the tangent vector of the meridian at $\mathbf{p}_0$, the parallel transport $\mathbf{v}'$ at $\mathbf{q}$ is tangent to meridian, so it points downward. Because parallel transport preserves angles, and $\mathbf{v}'$ is orthogonal to tangent vector of the parallel at $\mathbf{q}$, the parallel transport $\mathbf{v}''$ at $\mathbf{r}$ is also orthogonal to parallel. Finally, meridians are geodesics, thus parallel transport $\mathbf{v}'''$ at $\mathbf{p}_0$ is tangent to the meridian. Therefore the angle of $\mathbf{v}$ and $\mathbf{v}'''$ is θ.
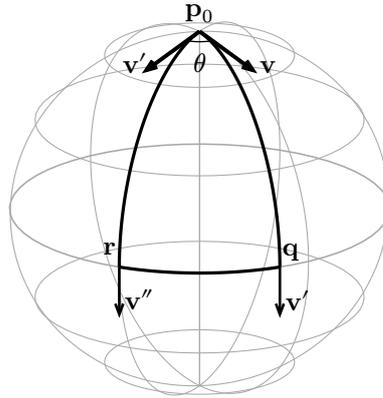
b. The parallel transport $\mathbf{v}'$ at $\mathbf{q}$ is tangent to the meridian at $\mathbf{q}$. Then we consider the cone which is tangent to the sphere along the parallel C of colatitude φ. We can determine the parallel transport of $\mathbf{v}'$ along the C relative to the tangent cone (see Example 1), and therefore the angle between the parallel transport $\mathbf{v}''$ at $\mathbf{r}$ and the tangent vector of the meridian at $\mathbf{r}$ is $\theta \cos \varphi$. Finally, since parallel transport preserves angles, the angle between the parallel transport $\mathbf{v}'''$ at $\mathbf{p}_0$ and the tangent vector of the meridian is also $\theta \cos \varphi$. Hence the angle between $\mathbf{v}$ and $\mathbf{v}'''$ is $\theta(1 - \cos \varphi)$.
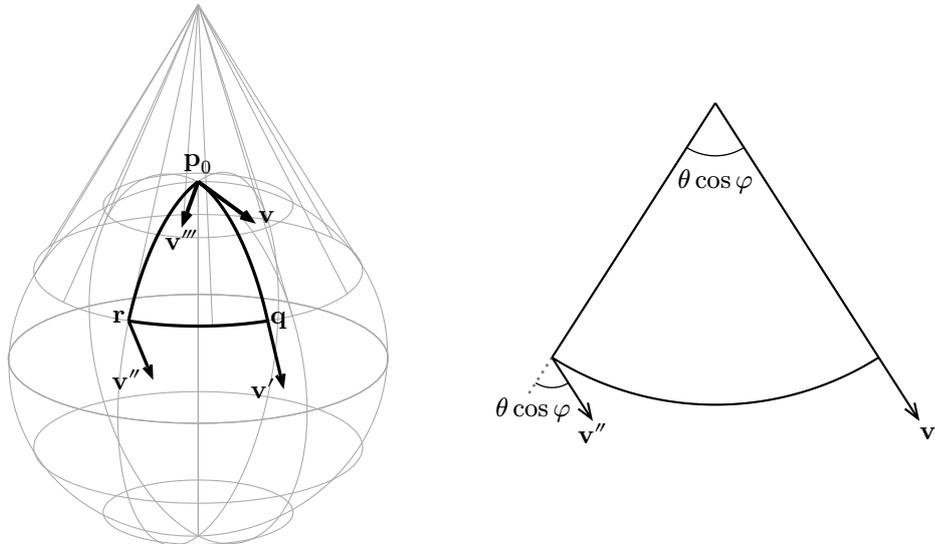
16. Since $\mathbf{p} \in S$ is a nonumbilical point of S, it is possible to parametrize a neighborhood of $\mathbf{p}$ by $\mathbf{x}(u, v)$ in such a way that the coordinate curves of this parametrization are the lines of curvature of S, and that $v = \text{const.}$ are the asymptotic curves. Then the fact that coordinates curves are the lines of curvature implies that $F = f = 0$. Note that $v = \text{const.}$ satisfies the differential equation of the asymptotic curves

$$e(u')^2 + 2fu'v' + g(v')^2 = 0,$$

it follows that $e = 0$ in the neighborhood of $\mathbf{p}$. Now recall the Mainardi-Codazzi equations

$$e_v = \frac{E_v}{2}\left(\frac{e}{E} + \frac{g}{G}\right) \implies 0 = \frac{E_v}{2}\left(0 + \frac{g}{G}\right).$$

Since $u = \text{const.}$ are not the asymptotic curves, it must be $g \neq 0$, therefore $E_v = 0$. Along the $v = \text{const.}$ curve, the normal curvature k_n is zero by definition, and the geodesic curvature is also zero because

$$k_g = -\frac{E_v}{2E\sqrt{G}} = 0$$

by Proposition 4. Therefore $k^2 = k_n^2 + k_g^2 = 0$ and that the asymptotic curve through $\mathbf{p}$ is an open segment of a straight line.

Note that the condition of having a neighborhood of parabolic points is essential, if not, there exists a counterexample like the upper parallel of the torus.

17. The parametrized surface $\mathbf{x} : U \times (-1,1) \to \mathbb{R}^3$ is regular because

$$\mathbf{x}_s \wedge \mathbf{x}_v = (\alpha' + v\mathbf{b}') \wedge \mathbf{b} = (\mathbf{t} + v\tau\mathbf{n}) \wedge \mathbf{b} = -\mathbf{n} + v\tau\mathbf{t} \neq 0,$$

so $d\mathbf{x_q}$ is one-to-one for all $\mathbf{q} \in U = I \times (-1,1)$. Therefore $\mathbf{x}(I \times (-\varepsilon, \varepsilon))$ is a regular surface for small enough $\varepsilon > 0$.

Now we assert that $\alpha(I)$ is a geodesic. Since the normal vector $\mathbf{n}$ of the curve and the normal vector $\mathbf{N} \parallel \mathbf{x}_s \wedge \mathbf{x}_v = -\mathbf{n}$ of the surface are parallel on the curve $\alpha(I)$, we conclude that $\alpha(I)$ is a geodesic.

[Unsolved] By Prop. 2 of Sec. 2-3, for each $\mathbf{q} \in U$, there exists a neighborhood V of $\mathbf{q}$ in $\mathbb{R}^2$ such that $\mathbf{x}(V) \subset \mathbb{R}^2$ is a regular surface. However, the existence of $\varepsilon > 0$ such that $x(I \times (-\varepsilon, \varepsilon))$ is a regular surface is not immediately evident. It remains unclear how to extend this local property to a global one over the entire interval I.

18. By Clairaut's relation, we know that $r\cos\theta$ is constant along the geodesic. Since $r\cos\theta = 1$ at the start point $\mathbf{p}$, we have $r\cos\theta = 1$ at every point along the geodesic. Since $\theta > 0$ when $r > 1$, the curve continues to point downwards and r decreases monotonically. As $r \to 1^+$, $\cos\theta \to 1^-$ and thus $\theta \to 0^+$. Consequently, the curve approaches asymptotically the parallel $x^2 + y^2 = 1$, $z = 0$.

19. Note that $E(u')^2 + 2Fu'v' + G(v')^2 = 1$, because the curve is parametrized by an arc length. Differentiating this expression to obtain

$$0 = (E_u u' + E_v v')(u')^2 + E \cdot 2u'u''$$
$$+ 2(F_u u' + F_v v')u'v' + 2F(u''v' + u'v'')$$
$$+ (G_u u' + G_v v')(v')^2 + G \cdot 2v'v'',$$

that is,

$$E_u(u')^3 + G_v(v')^3 + (E_v + 2F_u)(u')^2 v' + (G_u + 2F_v)u'(v')^2$$
$$+ 2Eu'u'' + 2Fv'u'' + 2Fu'v'' + 2Gv'v'' = 0.$$

Now define $\Psi_1 = u'' + \Gamma^1_{11}(u')^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2$ and $\Psi_2 = v'' + \Gamma^2_{11}(u')^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}(v')^2$, and recall that the Christoffel symbols satisfy the equations

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma^1_{11} \\ \Gamma^2_{11} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}E_u \\ F_u - \frac{1}{2}E_v \end{pmatrix}, \quad \begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma^1_{12} \\ \Gamma^2_{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}E_v \\ \frac{1}{2}G_u \end{pmatrix}, \quad \begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma^1_{22} \\ \Gamma^2_{22} \end{pmatrix} = \begin{pmatrix} F_v - \frac{1}{2}G_u \\ \frac{1}{2}G_v \end{pmatrix}.$$

Since we have

$$E\Psi_1 + F\Psi_2 = E\left(u'' + \Gamma^1_{11}(u')^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2\right) + F\left(v'' + \Gamma^2_{11}(u')^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}(v')^2\right)$$

$$= Eu'' + Fv'' + (E\Gamma^1_{11} + F\Gamma^2_{11})(u')^2 + 2(E\Gamma^1_{12} + F\Gamma^2_{12})u'v' + (E\Gamma^1_{22} + F\Gamma^2_{22})(v')^2$$

$$= Eu'' + Fv'' + \frac{1}{2}E_u(u')^2 + E_v u'v' + \left(F_v - \frac{1}{2}G_u\right)(v')^2.$$

Similarly, we obtain $F\Psi_1 + G\Psi_2 = Fu'' + Gv'' + \left(F_u - \frac{1}{2}E_v\right)(u')^2 + G_u u'v' + \frac{1}{2}G_v(v')^2$. Then

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$$2u'(E\Psi_1 + F\Psi_2) + 2v'(F\Psi_1 + G\Psi_2) = E_u(u')^3 + G_v(v')^3 + (E_v + 2F_u)(u')^2 v' + (G_u + 2F_v)u'(v')^2$$
$$+2Eu'u'' + 2Fv'u'' + 2Fu'v'' + 2Gv'v''$$
$$= 0,$$

thus we get $(Ee' + Fv')\Psi_1 + (Fu' + Gv')\Psi_2 = 0$. Therefore if the first equation $\Psi_1 = 0$ holds and $Fu' + Gv' \neq 0$, then we can conclude that the second equation $\Psi_2 = 0$ holds.

Note Since $Fu' + Gv' = \langle \mathbf{x}_u u' + \mathbf{x}_v v', \mathbf{x}_v \rangle = \langle \mathbf{t}, \mathbf{x}_v \rangle$, the condition $Fu' + Gv' \neq 0$ is not guaranteed by the condition that the curve is not a coordinate curve.

20. a. By Clairaut's relation, we know that $f \cos\theta$ is constant, where θ is the angle between the geodesic and a parallel. Since $f \cos\theta = (r\cos\pi/2 + a)\cos 0 = a$, we have

$$f\cos\theta = (r\cos u + a)\cos\theta = a$$

everywhere. Since $\cos\theta \leq 1$, it must be $r\cos u + a \geq a$, that is, $-\pi/2 \leq u \leq \pi/2$.

b. At the intersection point with the parallel $u = 0$, we have $f\cos\theta = (a + r)\cos\theta$. Note that f decreases on the geodesic until the intersection angle φ is positive, where it is determined by the equation $f\cos\varphi = (a + r)\cos\theta$. Since

$$\cos\varphi = \frac{a+r}{f(u)}\cos\theta \leq \frac{a+r}{a-r}\cos\theta < 1, \quad \text{for all } u \in [0, 2\pi),$$

we conclude that the geodesic also intersects the parallel $u = \pi$.

21. a. By simple calculation, the Christoffel symbols are given by

$$\Gamma_{11}^1 = \frac{U'}{2(U+V)}, \quad \Gamma_{12}^1 = \frac{V'}{2(U+V)}, \quad \Gamma_{22}^1 = -\frac{U'}{2(U+V)},$$

$$\Gamma_{11}^2 = -\frac{V'}{2(U+V)}, \quad \Gamma_{12}^2 = \frac{U'}{2(U+V)}, \quad \Gamma_{22}^2 = \frac{V'}{2(U+V)}.$$

Hence the differential equations of the geodesics is

$$u'' + \frac{U'}{2(U+V)}(u')^2 + \frac{V'}{U+V}u'v' - \frac{U'}{2(U+V)}(v')^2 = 0,$$

$$v'' - \frac{V'}{2(U+V)}(u')^2 + \frac{U'}{U+V}u'v' + \frac{V'}{2(U+V)}(v')^2 = 0.$$

Observe that by the first equation, we have

$$\frac{d}{ds}((U+V)u') = (U'u' + V'v')u' + (U+V)u''$$

$$= (U'u' + V'v')u' + \left(-\frac{1}{2}U'(u')^2 - V'u'v' + \frac{1}{2}U'(v')^2\right)$$

$$= \frac{1}{2}U'\left((u')^2 + (v')^2\right).$$

We multiply both sides by $2(U+V)u'$ in the equation above, then

$$\frac{d}{ds}((U+V)u')^2 = 2(U+V)u' \cdot \frac{d}{ds}((U+V)u') = 2(U+V)u' \cdot \frac{1}{2}U'\left((u')^2 + (v')^2\right)$$

$$= U'u'(U+V)\left((u')^2 + (v')^2\right) = \frac{dU}{ds}(U+V)\left((u')^2 + (v')^2\right).$$

Suppose that the curve is parametrized by an arc length, so that

$$E(u')^2 + 2Fu'v' + G(v')^2 = (U+V)\left((u')^2 + (v')^2\right) = 1.$$

Therefore we get

$$\frac{d}{ds}((U+V)u')^2 = \frac{dU}{ds} \implies ((U+V)u')^2 = U - c.$$

Similarly, by the second equation of the differential equations of the geodesics, we obtain

$$((U+V)v')^2 = V - c'.$$

By adding the two equations, since $((U+V)u')^2 + ((U+V)v')^2 = U + V$, it must be $c + c' = 0$. Thus

$$(U+V)u' = \pm\sqrt{U-c}, \ (U+V)v' = \pm\sqrt{V+c} \implies \frac{du}{\sqrt{U-c}} = \pm\frac{dv}{\sqrt{V+c}},$$

and the geodesics of a surface of Liouville may be obtained by integration in the form

$$\int \frac{du}{\sqrt{U-c}} = \pm \int \frac{dv}{\sqrt{V+c}} + c_1.$$

b. Observe that $\cos\theta$ and $\sin\theta$ satisfies the relation

$$\cos\theta = \frac{|\langle \mathbf{x}_u, \mathbf{x}_u u' + \mathbf{x}_v v' \rangle|}{|\mathbf{x}_u|} = \left|\sqrt{E}u'\right| = \sqrt{U+V}|u'|,$$

$$\sin\theta = \frac{|\langle \mathbf{x}_v, \mathbf{x}_u u' + \mathbf{x}_v v' \rangle|}{|\mathbf{x}_v|} = \left|\sqrt{G}v'\right| = \sqrt{U+V}|v'|.$$

Therefore

$$U\sin^2\theta - V\cos^2\theta = U(U+V)(u')^2 - V(U+V)(v')^2 = (U+V)\left(U(v')^2 - V(u')^2\right)$$

$$= (U+V)\left(U\frac{V+c}{(U+V)^2} - V\frac{U-c}{(U+V)^2}\right) = c = \text{const.}$$

22. We use the result from Exercise 15(a). Given rotation angle $\psi \in [0, 2\pi)$ of R, we can construct the piecewise regular curve α as following. Regard $\mathbf{p}$ as a pole, and let $\mathbf{q}, \mathbf{r}$ be two points on the equator in such a way that the meridians $\mathbf{pq}$ and $\mathbf{pr}$ make an angle ψ at $\mathbf{p}$. α consists of the meridian $\mathbf{pq}$, the parallel $\mathbf{qr}$, and the meridian $\mathbf{rp}$. Then by Exercise 15(a), it must be $R = P_\alpha$.

23. From Exercise 11(b) of Sec. 4-2, we already showed that restrictions to $\mathbb{S}^2$ of the linear orthogonal transformations of $\mathbb{R}^3$ is the isometry of the unit sphere. Now we show that every isometry are the restrictions to $\mathbb{S}^2$ of the linear orthogonal transformations. Let $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ be an isometry of $\mathbb{S}^2$. By Exercise 3 of Sec. 4-2, an arc length of any parametrized curve in $\mathbb{S}^2$ is equal to the arc length of the image curve by φ. Note that the intrinsic distance $d(\mathbf{p}, \mathbf{q})$ between two points $\mathbf{p}, \mathbf{q} \in \mathbb{S}^2$ is the angle between $\mathbf{p}$ and $\mathbf{q}$. It follows that the angle between $\mathbf{p}$ and $\mathbf{q}$ is same as the angle between $\varphi(\mathbf{p})$ and $\varphi(\mathbf{q})$, that is, $\langle \mathbf{p}, \mathbf{q} \rangle = \langle \varphi(\mathbf{p}), \varphi(\mathbf{q}) \rangle$. So if we define $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ as

$$\psi(\mathbf{p}) = \begin{cases} |\mathbf{p}|\varphi\left(\frac{\mathbf{p}}{|\mathbf{p}|}\right), & \mathbf{p} \neq \mathbf{0} \\ \mathbf{0}, & \mathbf{p} = \mathbf{0}, \end{cases}$$

then

$$\langle \psi(\mathbf{p}), \psi(\mathbf{q}) \rangle = \left\langle |\mathbf{p}|\varphi\left(\frac{\mathbf{p}}{|\mathbf{p}|}\right), |\mathbf{q}|\varphi\left(\frac{\mathbf{q}}{|\mathbf{q}|}\right) \right\rangle = |\mathbf{p}||\mathbf{q}|\left\langle \varphi\left(\frac{\mathbf{p}}{|\mathbf{p}|}\right), \varphi\left(\frac{\mathbf{q}}{|\mathbf{q}|}\right) \right\rangle = |\mathbf{p}||\mathbf{q}|\left\langle \frac{\mathbf{p}}{|\mathbf{p}|}, \frac{\mathbf{q}}{|\mathbf{q}|} \right\rangle = \langle \mathbf{p}, \mathbf{q} \rangle$$

for arbitrary $\mathbf{p}, \mathbf{q} \in \mathbb{R}^3 - \{\mathbf{0}\}$. Since $\psi(\mathbf{0}) = \mathbf{0}$, it is actually $\langle \psi(\mathbf{p}), \psi(\mathbf{q}) \rangle = \langle \mathbf{p}, \mathbf{q} \rangle$ for every $\mathbf{p}, \mathbf{q} \in \mathbb{R}^3$. Observe that

$$|\psi(\mathbf{p}) - \psi(\mathbf{q})|^2 = |\psi(\mathbf{p})|^2 + |\psi(\mathbf{q})|^2 - 2\langle \psi(\mathbf{p}), \psi(\mathbf{q}) \rangle = |\mathbf{p}|^2 + |\mathbf{q}|^2 - 2\langle \mathbf{p}, \mathbf{q} \rangle = |\mathbf{p} - \mathbf{q}|^2, \quad \mathbf{p}, \mathbf{q} \in \mathbb{S}^2.$$

Hence $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ is a distance-preserving map, and there exists $\mathbf{p}_0 \in \mathbb{R}^3$ and a linear isometry $\mathbf{F}$ of $\mathbb{R}^3$ such that $\psi(\mathbf{p}) = \mathbf{F}(\mathbf{p}) + \mathbf{p}_0$ by Exercise 8 of Sec. 4-2. Since φ is an isometry of $\mathbb{S}^2$, it must be $\mathbf{p}_0 = \mathbf{0}$. Therefore $\varphi = \mathbf{F}|_{\mathbb{S}^2}$ is the restriction to $\mathbb{S}^2$ of the linear orthogonal transformation $\mathbf{F}$ of $\mathbb{R}^3$.

4-5. The Gauss-Bonnet Theorem and Its Applications

1. Since the surface is compact, by Exercise 16 of Sec. 3-3, there is an elliptic point $\mathbf{p}_1$ in S so that $K(\mathbf{p}_1) > 0$. Now suppose that there is no point $\mathbf{p} \in S$ such that $K(\mathbf{p}) < 0$, that is, $K(\mathbf{p}) \geq 0$ for all $\mathbf{p} \in S$. Since $K(\mathbf{p}_1) > 0$, we have

$$0 < \iint_S K \, d\sigma = 2\pi\chi(S)$$

by Corollary 2. It follows that $\chi(S) = 2 = \chi(\mathbb{S}^2)$, thus S is homeomorphic to a sphere $\mathbb{S}^2$, which is a contradiction. Therefore there exists a point $\mathbf{p}_2$ such that $K(\mathbf{p}_2) < 0$. Finally, since S is connected, we can connect $\mathbf{p}_1$ and $\mathbf{p}_2$ by a regular curve, and $K(\mathbf{p}_3) = 0$ for some point $\mathbf{p}_3$ on that curve by the intermediate value theorem.

2. Recall that the torus is parametrized by

$$\mathbf{x}(u, v) = ((r\cos u + a)\cos v, (r\cos u + a)\sin v, r\sin u),$$

and

$$E = r^2, \quad F = 0, \quad G = (a + r\cos u)^2, \quad K = \frac{\cos u}{r(a + r\cos u)}.$$

Therefore we have

$$\iint_T K \, d\sigma = \int_0^{2\pi} \int_0^{2\pi} \frac{\cos u}{r(a + r\cos u)} \sqrt{r^2(a + r\cos u)^2} \, du \, dv$$

$$= \int_0^{2\pi} \int_0^{2\pi} \cos u \, du \, dv = 0.$$

Now we check the above result with the Gauss-Bonnet theorem. Since there is one handle in the torus, $\chi(T) = 0$. The torus is an oriented compact surface, we can apply Corollary 2 to torus and we obtain

$$\iint_T K \, d\sigma = 2\pi\chi(T) = 0.$$

3. Since S has positive Gaussian curvature, it is homeomorphic to a sphere. So the simple closed curve Γ divides S into two simple regions A and B, hence $\chi(A) = \chi(B) = 1$. Now we assert that

$$\text{area}(\mathbf{N}(A)) = \text{area}(\mathbf{N}(B)) = 2\pi.$$

We will apply Corollary 1 to A, whose boundary is Γ which has zero geodesic curvature and $\sum_i \theta_i = 0$. Therefore

$$\int_\Gamma k_g(s) \, ds + \iint_A K \, d\sigma + \sum_{i=0}^k \theta_i = \iint_A K \, d\sigma = 2\pi.$$

Note that if $\mathbf{x}(u, v)$ is a parametrization for S, then $\mathbf{N}_u = a_{11}\mathbf{x}_u + a_{12}\mathbf{x}_v$, $\mathbf{N}_v = a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v$ and

$$\mathbf{N}_u \wedge \mathbf{N}_v = (a_{11}\mathbf{x}_u + a_{12}\mathbf{x}_v) \wedge (a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v) = (a_{11}a_{22} - a_{12}a_{21})(\mathbf{x}_u \wedge \mathbf{x}_v) = K(\mathbf{x}_u \wedge \mathbf{x}_v).$$

By Prop. 2, there is a triangulation of $\mathfrak{J}$ of A such that every triangle $T_i \in \mathfrak{J}$ is contained in some coordinate neighborhood of the family $\{\mathbf{x}^i\}$. Hence

$$\text{area}(\mathbf{N}(A)) = \sum_i \iint_{(\mathbf{N}^i \circ \mathbf{x}^i)^{-1}(\mathbf{N}^i(T_i))} \left|\mathbf{N}_u^i \wedge \mathbf{N}_v^i\right| \, du \, dv$$

$$= \sum_i \iint_{(\mathbf{x}^i)^{-1}(T_i)} |K|\left|\mathbf{x}_u^i \wedge \mathbf{x}_v^i\right| \, du \, dv$$

$$= \sum_i \iint_{T_i} K \, d\sigma = \iint_A K \, d\sigma = 2\pi.$$

We conclude that $\mathbf{N}(A)$ and $\mathbf{N}(B)$ have the same area 2π.

4. **a.** By Exercise 4 of Sec. 2-3, an ellipsoid S is diffeomorphic to the sphere $\mathbb{S}^2$. Therefore we obtain $\chi(S) = \chi(\mathbb{S}^2) = 2$ by Prop. 4.

b. Observe that the map $\varphi : S \to \mathbb{S}^2$ defined by $\varphi(x, y, z) = (x, y^5, z^3)$ is a homeomorphism. Therefore $\chi(S) = \chi(\mathbb{S}^2) = 2$ by Prop. 4.

5. By the argument for the interpretation of K (p. 274), we have

$$\Delta\varphi = \iint_R K \, d\sigma.$$

Since $K \equiv 1$ on the sphere, we have

$$\Delta\varphi = \iint_R K \, d\sigma = \iint_R d\sigma = \int_0^{2\pi} \int_0^\varphi \sin\varphi \, d\varphi \, d\theta = 2\pi(1 - \cos\varphi).$$

Since $\Delta\varphi$ is the same as A, which is the area of the region R of $\mathbb{S}^2$ bounded by C, we also have that

$$\lim_{A\to 0} \frac{\Delta\varphi}{A} = \lim_{A\to 0} 1 = 1 = \text{curvature of } \mathbb{S}^2.$$

6. To compute the index, we will restrict the vector field $\mathbf{v}$ to the curve $\alpha(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. If $\varphi(t)$ is the angle that $\mathbf{v}(t)$ forms with the x axis, then we have $2\pi I = \varphi(2\pi) - \varphi(0)$.

   a. $\mathbf{v}(x, y) = (x, y) = (0, 0)$ if and only if $x = 0$ and $y = 0$. So $(0, 0)$ is an isolated singular point. Since $\mathbf{v}(t) = (\cos t, \sin t)$, we have $\varphi(t) = t$ and $2\pi I = 2\pi \implies I = 1$.

   b. $\mathbf{v}(x, y) = (-x, y) = (0, 0)$ if and only if $x = 0$ and $y = 0$, so $(0, 0)$ is an isolated singular point. Since $\mathbf{v}(t) = (-\cos t, \sin t) = (\cos(\pi - t), \sin(\pi - t))$, we have $\varphi(t) = \pi - t$ and $2\pi I = (-\pi) - \pi = -2\pi$, thus $I = -1$.

   c. $\mathbf{v}(x, y) = (x, -y) = (0, 0)$ if and only if $x = 0$ and $y = 0$, so $(0, 0)$ is an isolated singular point. Since $\mathbf{v}(t) = (\cos t, -\sin t) = (\cos(-t), \sin(-t))$, we have $\varphi(t) = -t$ and $2\pi I = (-2\pi) - 0 = -2\pi$, thus $I = -1$.

   d. $\mathbf{v}(x, y) = (x^2 - y^2, -2xy) = (0, 0)$ implies $x = \pm y$ and $xy = 0$, thus $x = y = 0$. Hence $(0, 0)$ is an isolated singular point. Observe that

   $$\mathbf{v}(t) = (\cos^2 t - \sin^2 t, -2\cos t \sin t) = (\cos 2t, -\sin 2t) = (\cos(-2t), \sin(-2t)),$$

   Thus $\varphi(t) = -2t$. Therefore $2\pi I = (-4\pi) - 0 = -4\pi$ and $I = -2$.

   e. Note that $(x - yi)^3 = (x^3 - 3x^2 y) + (y^3 - 3x^2 y)i$. Thus $\mathbf{v}(x, y) = (x^3 - 3xy^2, y^3 - 3x^2 y) = (0, 0)$ implies $(x - iy)^3 = 0$, and $x = y = 0$. So $(0, 0)$ is an isolated singular point. Since

   $$\mathbf{v}(t) = (\cos^3 t - 3\cos^2 t \sin t, \sin^3 t - 3\cos^2 t \sin t) = (\cos(-3t), \sin(-3t)),$$

   we have $\varphi(t) = -3t$ and $2\pi I = (-6\pi) - 0 \implies I = -3$.

7. Yes. Consider the vector field $\mathbf{v}(x, y) = (x^2 + y^2, x^2 + y^2)$. Then $(0, 0)$ is an isolated singular point. We restrict $\mathbf{v}$ to the curve $\alpha(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Then the angle $\varphi(t)$ that $\mathbf{v}(t) = (1, 1)$ forms with the x axis is $\pi/4$. Therefore we have $2\pi I = \varphi(2\pi) - \varphi(0) = 0$, $I = 0$.

8. First suppose that an oriented compact surface $S \subset \mathbb{R}^3$ has a differentiable vector field without singular points. Then

$$\chi(S) = \frac{1}{2\pi} \iint_S K \, d\sigma = \sum_{i=1}^k I_i = 0.$$

(In fact, there are no singular points, so $k = 0$.) We have $\chi(S) = 0 = \chi(T)$ and S is homeomorphic to a torus.

Conversely, suppose that S is homeomorphic to a torus. Recall that torus can be parametrized by

$$\mathbf{x}(u, v) = ((r\cos u + a)\cos v, (r\cos u + a)\sin v, r\sin u), \quad u \in (0, 2\pi), \ v \in (0, 2\pi).$$

Then the vector field $\mathbf{w}(u, v) = \mathbf{x}_v(u, v) = (-(r\cos u + a)\sin v, (r\cos u + a)\cos v, 0)$ is differentiable and has size $|r\cos u + a| > a - r > 0$. Therefore there is a differentiable vector field without singular points on the torus. Since there is a diffeomorphism $\varphi : T \to S$ between a torus T and the surface S, the mapping $d\varphi(\mathbf{w})$ is the differentiable vector field on S, and this doesn't have singular points because $d\varphi(\mathbf{w}) = 0 \iff \mathbf{w} = 0$.

9. A regular closed simple curve C divides $\mathbb{S}^2$ into two disjoint open regions, say R_1 and R_2, which are homeomorphic to an open disk. Let $\alpha : I \to C$ be a parametrization by an arc length of C such that

$\alpha(0) = \alpha(l)$. For tangent vector $\mathbf{t}(s) = \alpha'(s)$, there exists an orthonormal basis $\{\mathbf{t}(s), \mathbf{n}(s)\}$ of $T_{\alpha(s)}(\mathbb{S}^2)$. Then we can write

$$\mathbf{v}(s) = a(s)\mathbf{t}(s) + b(s)\mathbf{n}(s)$$

for some differentiable functions $a(s)$ and $b(s)$. Since the trajectories of $\mathbf{v}$ are never tangent to C, we have $b(s) \neq 0$ for all s. So it follows that $b(s)$ has fixed sign, and that $\mathbf{v}$ always points inward toward R_1 or outward toward R_1 at every point on C. Let $\theta_{\mathbf{t}}(s)$ be an angle from $\mathbf{x}_\theta(\alpha(s))$ (the parallel) to $\mathbf{t}(s)$, and $\theta_{\mathbf{v}}(s)$ be an angle from $\mathbf{x}_\theta(\alpha(s))$ to $\mathbf{v}(s)$. Then the angular difference $\varphi(s) = \theta_{\mathbf{v}}(s) - \theta_{\mathbf{t}}(s)$ between $\theta_{\mathbf{v}}(s)$ and $\theta_{\mathbf{t}}(s)$ is confined within the open interval $(0, \pi)$ or $(\pi, 2\pi)$. Since $\varphi(l) - \varphi(0)$ must be an integer multiple of 2π, we have $\varphi(l) - \varphi(0) = 0$. Therefore

$$2\pi I_{\mathbf{v}} = \theta_{\mathbf{v}}(l) - \theta_{\mathbf{v}}(0) = \theta_{\mathbf{t}}(l) - \theta_{\mathbf{t}}(0),$$

where $I_{\mathbf{v}}$ is the index of $\mathbf{v}$ along $C \subset \overline{R_1}$, respectively. By Hopf's theorem, we have $\theta_{\mathbf{t}}(l) - \theta_{\mathbf{t}}(0) = \pm 2\pi$. hence $I_{\mathbf{v}} = \pm 1$, and therefore R_1 contains at least one singular point of $\mathbf{v}$. Similarly, R_2 also contains at least one singular point of $\mathbf{v}$.