2025 적분 챔피언십: 최종진출전

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최종진출전

* 참가자 수 변경으로 인해 6번 문제부터 출제되었습니다.

문제 6

$$\int \left(\frac{1}{\log x} - \frac{1}{(\log x)^2} \right) dx$$

풀이. $(x/(\log x))' = 1/(\log x) - 1/(\log x)^2$ 이므로

$$I = \boxed{\frac{x}{\log x}}$$

문제 7

$$\int \log \left(1 + \frac{1}{x}\right)^x dx$$

풀이. 부분적분 하면

$$\begin{aligned} I &= \int x \log\left(1 + \frac{1}{x}\right) dx \\ &= \frac{1}{2}x^2 \log\left(1 + \frac{1}{x}\right) - \int \left(-\frac{1}{x^2 + x}\right) \cdot \frac{1}{2}x^2 \, dx \\ &= \boxed{\frac{1}{2}x^2 \log\left(1 + \frac{1}{x}\right) + \frac{x}{2} - \frac{1}{x}\log(x+1)} \end{aligned}$$

문제 8

$$\int_0^1 \cos(1 + \log(x)) \, dx$$

풀이. $t = 1 + \log(x)$로 치환하면

$$I = \int_{-\infty}^{1} e^{t-1} \cos t \, dt = \frac{1}{e}\left[\frac{1}{2}e^t(\cos t + \sin t)\right]_{-\infty}^{1} = \boxed{\frac{\cos(1) + \sin(1)}{2}}$$

문제 9

$$\sum_{n=2}^{\infty}\int_0^{\infty}\frac{x^2}{1+2nx^2+(n^2-1)x^4}\,dx$$

풀이.

$$\begin{aligned}\sum_{n=2}^{\infty}\int_0^{\infty}\frac{x^2}{1+2nx^2+(n^2-1)x^4}\,dx &= \sum_{n=2}^{\infty}\int_0^{\infty}\frac{1}{2}\left(\frac{1}{1+(n-1)x^2}-\frac{1}{1+(n+1)x^2}\right)dx \\ &= \sum_{n=2}^{\infty}\frac{1}{2}\left(\frac{\pi}{2\sqrt{n-1}}-\frac{\pi}{2\sqrt{n+1}}\right) \\ &= \frac{\pi}{4}\left(1+\frac{1}{\sqrt{2}}\right) = \boxed{\frac{\left(2+\sqrt{2}\right)\pi}{8}}\end{aligned}$$

문제 10

$$\int \sec^2(x)(1+2\tan(x))e^x\,dx$$

풀이. $(\sec^2(x)e^x)' = \sec^2(x)e^x + 2\sec(x)\cdot 2\sec(x)e^x\cdot\sec(x)\tan(x) = \sec^2(x)(1+2\tan(x))e^x$ 이다.

$$I = \boxed{\sec^2(x)e^x}$$

문제 11

$$\int\left(x-\frac{1}{x^3}\right)\cosh\left(\frac{1}{x}\right)e^x\,dx$$

풀이.

$$\begin{aligned}I &= \frac{1}{2}\int\left(x-\frac{1}{x^3}\right)\left(e^{x+\frac{1}{x}}+e^{x-\frac{1}{x}}\right)dx \\ &= \frac{1}{2}\left(\int\left(1-\frac{1}{x^2}\right)\left(x+\frac{1}{x}\right)e^{x+\frac{1}{x}}\,dx+\int\left(1+\frac{1}{x^2}\right)\left(x-\frac{1}{x}\right)e^{x-\frac{1}{x}}\,dx\right) \\ &= \boxed{\frac{1}{2}\left(\left(x+\frac{1}{x}-1\right)e^{x+\frac{1}{x}}+\left(x-\frac{1}{x}-1\right)e^{x-\frac{1}{x}}\right)}\end{aligned}$$

문제 12

$$\int\frac{e^x(x\cos x+(x-1)\sin x)}{x^2}\,dx$$

풀이.

$$\left(\frac{e^x \sin x}{x}\right)' = \frac{e^x(\sin x + \cos x)\cdot x - e^x \sin x}{x^2} = \frac{e^x(x\cos x + (x-1)\sin x)}{x^2}$$

이므로

$$I = \boxed{\frac{e^x \sin x}{x}}$$

문제 13

$$\int \frac{\arctan x}{(x^2+1)^2}\,dx$$

풀이. $t = \arctan x$로 치환하면

$$\begin{aligned} I &= \int t\cos^2 t\,dt = \frac{1}{2}\int t(1+\cos 2t)\,dt \\ &= \frac{1}{4}t^2 + \frac{1}{4}t\sin 2t + \frac{1}{8}\cos 2t = \boxed{\frac{1}{4}\arctan^2(x) + \frac{x\arctan(x)}{2(x^2+1)} + \frac{1-x^2}{8(x^2+1)}} \end{aligned}$$

문제 14

$$\int \sqrt{e^x+1}\,dx$$

풀이. $t = \sqrt{e^x+1}$로 치환하면

$$\begin{aligned} I &= \int \frac{2t^2}{t^2-1}\,dt = \int \left(2 + \frac{1}{t-1} - \frac{1}{t+1}\right)dt \\ &= 2t + \log\left(\frac{t-1}{t+1}\right) = \boxed{2\sqrt{e^x+1} + \log\left(\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\right)} \end{aligned}$$

문제 15

$$\int_0^{2025\pi} \left\lfloor \frac{3}{2} + \sin x \right\rfloor dx$$

풀이.

$$\left\lfloor \frac{3}{2} + \sin x \right\rfloor = \begin{cases} 0, & \text{when } \sin x < -1/2 \\ 1, & \text{when } -1/2 \le \sin x < 1/2 \\ 2, & \text{when } \sin x \ge 1/2 \end{cases}$$

이다. $\sin x$의 주기성에 의해

$$
\begin{aligned}
I &= 1012 \cdot \int_0^{2\pi} \left\lfloor \frac{3}{2} + \sin x \right\rfloor dx + \int_0^{\pi} \left\lfloor \frac{3}{2} + \sin x \right\rfloor dx \\
&= 1012\left(2 \cdot \frac{2\pi}{3} + 1 \cdot \frac{2\pi}{3} + 0 \cdot \frac{2\pi}{3}\right) + \left(2 \cdot \frac{2\pi}{3} + 1 \cdot \frac{\pi}{3}\right) = \boxed{\frac{6077}{3}\pi}
\end{aligned}
$$

문제 16

$$
\int_0^1 \sqrt{1-x^2}\,\arcsin(x)\,dx
$$

풀이. $t = \arcsin(x)$로 치환하면

$$
\begin{aligned}
I &= \int_0^{\pi/2} t\cos^2 t\,dt = \frac{1}{2}\int_0^{\pi/2} t(1+\cos 2t)\,dt \\
&= \left[\frac{1}{4}t^2 + \frac{1}{4}t\sin 2t + \frac{1}{8}\cos 2t\right]_0^{\pi/2} = \boxed{\frac{\pi^2}{16} - \frac{1}{4}}
\end{aligned}
$$

문제 17

$$
\int \frac{1}{1+20\sin x + 25\cos x}\,dx
$$

풀이. 바이어슈트라스 치환 $t = \tan(x/2)$를 사용하면

$$
\begin{aligned}
I &= \int \frac{1}{1 + 20\cdot\frac{2t}{t^2+1} + 25\cdot\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,dt = \int \frac{2}{-24t^2+40t+26}\,dt \\
&= -\frac{1}{12}\int \frac{1}{(t+\frac{1}{2})(t-\frac{13}{6})}\,dt = \frac{1}{32}\int\left(\frac{1}{t-\frac{13}{6}} - \frac{1}{t+\frac{1}{2}}\right)dt \\
&= \frac{1}{32}\log\left(\frac{t+\frac{1}{2}}{t-\frac{13}{6}}\right) \overset{+C}{=} \boxed{\frac{1}{32}\log\left(\frac{2\tan(x/2)+1}{6\tan(x/2)-13}\right)}
\end{aligned}
$$

문제 18

$$
\int_{-\infty}^{\infty} 2^x 3^{-4^x}\,dx
$$

풀이. $t = 2^x$로 치환하면

$$
I = \frac{1}{\log 2}\int_0^{\infty} e^{-(\log 3)t^2}\,dt = \boxed{\frac{1}{2\log 2}\sqrt{\frac{\pi}{\log 3}}}
$$

문제 19

$$\int \frac{1}{\sin^4 x + \cos^4 x}\, dx$$

풀이. $\sin^4(x) + \cos^4(x) = \left(\sin^2 x + \cos^2 x\right)^2 - 2\sin^2 x \cos^2 x = 1 - \frac{1}{2}\sin^2 2x = 4/(3 + \cos 4x)$ 이므로

$$I = \int \frac{4}{3 + \cos 4x}\, dx$$

바이어슈트라스 치환 $t = \tan(2x)$를 사용하면

$$I = \int \frac{4}{3 + \frac{1-t^2}{1+t^2}} \frac{1}{2(1+t^2)}\, dt = \int \frac{1}{t^2+2}\, dt = \frac{1}{\sqrt{2}} \arctan\left(\frac{t}{\sqrt{2}}\right) = \boxed{\frac{1}{\sqrt{2}} \arctan\left(\frac{\tan 2x}{\sqrt{2}}\right)}$$

문제 20

$$\int_0^{2025} \frac{\log(2025 - x)}{\sqrt{2025x - x^2}}\, dx$$

풀이. $x = 2025 \sin^2 t$로 치환하면

$$I = \int_0^{\pi/2} \frac{\log(2025 \cos^2 t)}{2025 \sin t \cos t} \cdot 2 \cdot 2025 \sin t \cos t\, dt = 2\int_0^{\pi/2} (\log 2025 + 2 \log \cos t)\, dt$$

이다. 이 때 $\int_0^{\pi/2} \log \cos t\, dt = -\frac{\pi}{2} \log 2$ 이므로

$$I = 2\left(\frac{\pi}{2} \log 2025 - \pi \log 2\right) = \pi \log \frac{2025}{4} = \boxed{2\pi \log \frac{45}{2}}$$